ผลเฉลยของสมการความร้อนโดยวิธีเอ็กซอดัส

The Solution of the Heat Equation by the Exodus Method

สมชาติ ฉันทศิริวรรณ
อาจารย์

ภาควิชาวิศวกรรมเครื่องกล
ม. เทคโนโลยีพระจอมเกล้าธนบุรี
กรุงเทพ ฯ 10140

The Exodus method of solving the general linear heat conduction problem is presented in this paper. Such a problem may have space-dependent conductivity, source term that is a linear function of temperature, time-dependent boundary conditions of the first kind, the second kind, and the third kind. A mathematical proof of solution method is also provided. The advantage of the Exodus method over a traditional method such as the Jacobi method lies in its ability to determine solution at specific location and time without having to determine the solution of the entire domain since the dependence of the solution on the initial condition, the boundary condition, and the source term is made explicit.

บทความนี้นำเสนอวิธีเอ็กซอดัส (Exodus method) สำหรับแก้ปัญหาเชิงเส้นของการนำความร้อนทั่วไป โดยปัญหาดังกล่าวอาจจะมีค่าการนำความร้อนที่ขึ้นกับตำแหน่ง มีแหล่งต้นทางความร้อนที่เป็นสมการเชิงเส้นของอุณหภูมิ และมีเงื่อนไขขอบที่ขึ้นอยู่กับเวลาทั้งประเภทที่หนึ่ง ประเภทที่สอง และประเภทที่สาม บทความนี้ยังแสดงบทพิสูจน์ความถูกต้องของผลเฉลยของปัญหาที่ได้มาโดยวิธีเอ็กซอดัสอีกด้วย ข้อได้เปรียบของวิธีเอ็กซอดัสคือความสามารถหาผลเฉลยที่ตำแหน่งใดตำแหน่งหนึ่งในโดเมนของปัญหาโดยไม่จำเป็นต้องทราบผลเฉลยของตำแหน่งอื่นๆ ทั้งนี้เนื่องจากผลเฉลยของปัญหาที่ได้มาโดยวิธีเอ็กซอดัส แสดงออกมาในรูปของเงื่อนไขเริ่มต้น เงื่อนไขขอบ และแหล่งต้นทางความร้อนอย่างชัดแจ้ง

1. Introduction

The Exodus method was first introduced by Emery and Carson (1966). It has been used to solve heat conduction problems (Naraghi and Tsai, 1993) and other engineering problems (Sadiku and Hunt, 1992; Sadiku, Ajose, and Fu, 1994). This method has its root in the Monte Carlo method (Haji-Sheikh and Sparrow, 1965), which approximates the heat diffusion process by a random walk process. Like the Monte Carlo method, the Exodus method can determine the solution at a few internal points in the domain without having to obtain the solution of the entire domain. However, the Exodus method does not require a random number generator as the Monte Carlo method does, and produces a more accurate solution than the Monte Carlo method.

So far, most of the problems that have been solved by the Exodus are steady-state problems (Naraghi and Tsai, 1993; Sadiku and Hunt, 1992; Sadiku, Ajose, and Fu, 1994; Zinsmeister and Sawyerr, 1974). In this paper, the Exodus method will be extended to deal with time-dependent problems. The problem to be considered describes the heat conduction in a material of which conductivity depends on location in space. There may be a source term that depends linearly on temperature. The boundary conditions of the problem may be of the first kind, the second kind, and the third kind.

2. Statement of the Problem

The governing equation describing heat conduction in an orthotropic material in the Cartesian coordinate is

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(\vec{r})\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa(\vec{r})\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa(\vec{r})\frac{\partial T}{\partial z}\right) + \xi(\vec{r})T + s(\vec{r},t) \qquad (1)$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Let the initial and boundary conditions be

$$T(\vec{r},0) = g(\vec{r}) \qquad (2)$$

$$T(\vec{r},t) = u(\vec{r},t) \quad \text{on } \Gamma_1 \qquad (3)$$

$$\kappa(\vec{r})\frac{\partial T}{\partial n} = -q(\vec{r},t) \quad \text{on } \Gamma_2 \qquad (4)$$

$$\kappa(\vec{r})\frac{\partial T}{\partial n} = h(\vec{r})\left(u(\vec{r},t) - T(\vec{r},t)\right) \quad \text{on } \Gamma_3 \qquad (5)$$

where is Γ_1 is the boundary of the first kind, Γ_2 is the boundary of the second kind, and Γ_3 is the boundary of the third kind. A numerical solution of Eqs. (1) - (5) may start with the construction of a structured grid in the solution domain Ω, resulting in N total interior and boundary nodes. Assume that the interior nodes in Ω are numbered from 1 to N_1, the boundary nodes on Γ_1 are numbered from $N_1 + 1$ to $N_1 + N_2$, the boundary nodes on Γ_2 are numbered from $N_1 + N_2 + 1$ to $N_1 + N_2 + N_3$, and the boundary nodes on Γ_3 are numbered from $N_1 + N_2 + N_3 + 1$ to N. The finite difference method can then be used to discretize Eqs. (1) - (5) to obtain the following systems of algebraic equations.

$$T_i^{l+1} = \sum_{j=1}^{N} a_{ij} T_j^l + \xi_i T_i^l \Delta t + s_i^l \Delta t, \quad i = 1, 2, \ldots, N_1 \qquad (6)$$

$$T_i^0 = g_i, \quad i = 1, 2, \ldots, N_1 \qquad (7)$$

$$T_i^l = f_i^l, \quad i = N_1 + 1, N_1 + 2, \ldots, N_1 + N_2 \qquad (8)$$

$$T_i^l = \sum_{j=1}^{N_1} b_{ij} T_j^l - q_i^l \Delta n_i, \quad i = N_1 + N_2 + 1, \ldots, N_1 + N_2 + N_3 \qquad (9)$$

$$T_i^l = \frac{1}{1 + h_i \Delta n_i}\left(\sum_{j=1}^{N_1} c_{ij} T_j^l + h_i \Delta n_i u_i^l \right), \quad i = N_1 + N_2 + N_3 + 1, \ldots, N \qquad (10)$$

where subscripts denote space indices and superscripts denote time indices. Note that the time derivative in Eq. (1) is being approximated by the forward time differencing scheme, making Eq. (6) accurate to $O(\Delta t)$. Equation (6) can be rewritten in the following alternative form, which will be useful later.

$$u_i^{l+1} = \sum_{j=1}^{N} p_{ij} u_j^l + s_i^l \Delta t, \quad i = 1, 2, \ldots, N_1 \qquad (11)$$

$$p_{ij} = a_{ij} + \delta_{ij} \xi_j \Delta t \quad i = 1, 2, \ldots, N_1 \qquad (12)$$

where δ_{ij} is the Kronecker delta.

3. Exodus Solution

The Exodus solution to Eq. (11) subjected to conditions prescribed by Eqs. (7) - (10) will be expressed in terms of Exodus function, v_{ij}^m. The characteristics of the Exodus function are the following.

(1) v_{ij}^m is a function of time level m, origin node i, and destination node j.

(2) If i is an interior node, and j is a boundary node,

$$v_{ij}^1 = p_{ij} \qquad (13)$$

(3) If i and j are interior nodes,

$$v_{ij}^1 = p_{ij} + \sum_{k=N_1+N_2+1}^{N_1+N_2+N_3} p_{ik} b_{kj} + \sum_{k=N_1+N_2+N_3+1}^{N} p_{ik} \frac{c_{kj}}{1 + h_k \Delta n_k} \qquad (14)$$

(4) For m > 0,

$$v_{ij}^{m+1} = \sum_{k=1}^{N_1} v_{ik}^1 v_{kj}^m \qquad (15)$$

Since only v_{ij}^m for i being an interior node is needed, Eqs. (13) - (15) are for determining v_{ij}^m uniquely for all i's and m's. An efficient method for finding v_{ij}^m is the Exodus method (Emery and Carson, 1966). Figure 1 shows a flow chart of a subroutine to implement this method. It should be noted that when ξ in Eq. (1) is zero, v_{ij}^m can be considered as the transition probability for a random walk starting at node i and reaching node j in time $m\Delta t$.

Once all necessary v_{ij}^m's are known, the following Exodus solution can be used to obtain temperature T_i^l at location i and time $l\Delta t$.

$$T_i^l = \sum_{j=1}^{N_1} v_{ij}^l g_j + \sum_{j=N_1+1}^{N_1+N_2}\left(\sum_{m=1}^{l} v_{ij}^m f_j^{l-m} \right) - \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3}\left(\sum_{m=1}^{l} v_{ij}^m q_j^{l-m} \Delta n_j \right) + \sum_{j=N_1+N_2+N_3+1}^{N}\left(\sum_{m=1}^{l} v_{ij}^m \frac{h_j \Delta n_j}{(1 + h_j \Delta n_j)} u_j^{l-m} \right) + \sum_{j=1}^{N_1}\left(\sum_{m=1}^{l} v_{ij}^{m-1} s_j^{l-m} \Delta t \right) \qquad (16)$$

4. Mathematical Proof of the Solution

From Eq. (11),

$$T_i^{l+1} = \sum_{j=1}^{N} p_{ij} T_j^l + s_i^l \Delta t$$

$$= \sum_{j=1}^{N_1} p_{ij} T_j^l + \sum_{j=N_1+1}^{N_1+N_2} p_{ij} T_j^l + \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} p_{ij} T_j^l + \sum_{j=N_1+N_2+N_3+1}^{N} p_{ij} T_j^l + s_i^l \Delta t$$

Substitute Eqs. (8), (9), and (10) into the above equation.

$$T_i^{l+1} = \sum_{j=1}^{N_1} p_{ij} T_j^l + \sum_{j=N_1+1}^{N_1+N_2} p_{ij} f_j^l + \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3}\left(\sum_{k=1}^{N_1} p_{ij} b_{jk} T_k^l \right) - \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} p_{ij} q_j^l \Delta n_j + \sum_{j=N_1+N_2+N_3+1}^{N}\left(\sum_{k=1}^{N_1}\left(\frac{p_{ij}}{1 - h_j \Delta n_j} \right) c_{jk} T_k^l \right) + \sum_{j=N_1+N_2+N_3+1}^{N}\left(\frac{p_{ij} h_j \Delta n_j}{1 + h_j \Delta n_j} \right) u_j^l + s_i^l \Delta t$$

Rearrange the equation, and make use of Eqs. (13) and (14).

$$T_i^{l+1} = \sum_{k=1}^{N_1} v_{ik}^1 T_k^l + \sum_{j=N_1+1}^{N_1+N_2} p_{ij} f_j^l - \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} p_{ij} q_j^l \Delta n_j + \sum_{j=N_1+N_2+N_3+1}^{N}\left(\frac{p_{ij} h_j \Delta n_j}{1 + h_j \Delta n_j} \right) u_j^l + s_i^l \Delta t$$

Substitute T_i^l from Eq. (16), rearrange the result, and make use of Eq. (15).

$$
\begin{aligned}
T_i^{l+1} = & \sum_{j=1}^{N_1} v_{ij}^{l+1} g_j + \sum_{j=N_1+1}^{N_1+N_2}\left(\sum_{m=1}^{l+1} v_{ij}^{m} f_j^{l+1-m}\right) - \\
& \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3}\left(\sum_{m=1}^{l+1} v_{ij}^{m} q_j^{l+1-m}\Delta n_j\right) + \\
& \sum_{j=N_1+N_2+N_3+1}^{N}\left(\sum_{m=1}^{l+1} v_{ij}^{m} \frac{h_j \Delta n_j}{\left(1+h_j\Delta n_j\right)} u_j^{l+1-m}\right) + \\
& \sum_{j=1}^{N_1}\left(\sum_{m=1}^{l+1} v_{ij}^{m-1} s_j^{l+1-m}\Delta t\right)
\end{aligned}
$$

which is Eq. (16) with l replaced by l + 1.

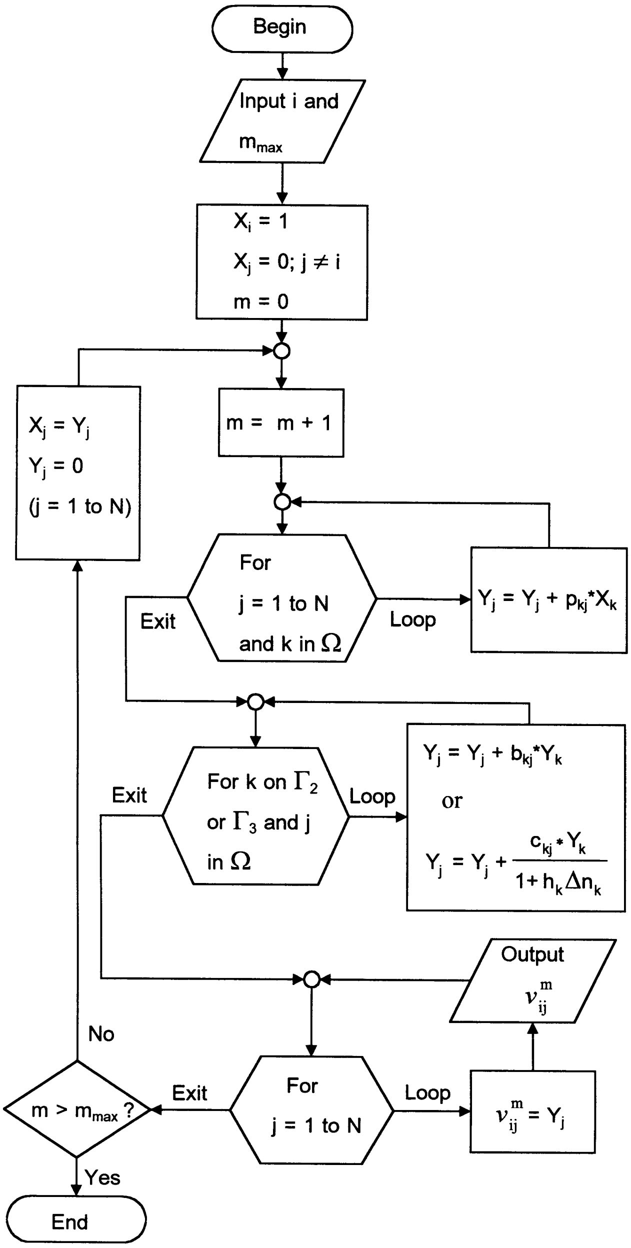

Fig. 1: Flow chart of an algorithm to compute Exodus functions

5. Conclusion

Equation (16) relates the solution at specified location and time to the source term, the initial condition, and the boundary condition. Hence, the Exodus method appears to be computationally efficient when the solution at isolated locations, instead of the solution over the entire domain, is desired. Although ξ in Eq. (1) and h in Eq. (5) are limited to functions of $\vec{r}$, many problems can still be dealt with by the Exodus method. It should also be noted that the Exodus method requires the explicit formulation of the discretized equations (6) - (10). This means that the maximum time step Δt is imposed by the formulation lest instability will result. The maximum time step may be very small, and excessive computation may be required. However, by choosing a more accurate formulation, it should be possible to use a larger grid size, which will lead to a larger allowable time step.

An example of a phenomenon described by Eq. (6) - (10) is heat conduction in a solid having space-dependent thermal conductivity. In this case, Eqs. (6) - (10) may be viewed as describing a linear system that has temperature at interior location i and time $l\Delta t$ as the output, and the nonhomogeneities of the system, which are the initial temperatures at interior mesh points, the source terms at interior mesh points, the boundary temperatures at mesh points on Γ_1, the boundary fluxes at mesh points on Γ_2, and the ambient temperatures at mesh points on Γ_3, as the inputs. By relating the output to the inputs, the Exodus solution can then be considered as a convenient method of calculating transfer functions for this particular linear system.

6. Nomenclature

a_{ij}, b_{ij}, c_{ij}	discretization coefficients
f	boundary temperature
g	initial temperature
h	heat transfer coefficient
l, m	numbers of time steps
n	space coordinate normal to the boundary
N	total number of nodes
N_1	number of interior nodes
N_2	number of boundary nodes on Γ_1
N_3	number of boundary nodes on Γ_2
N_4	number of boundary nodes on Γ_3
p_{ij}	discretization coefficients
q	heat flux
$\vec{r}$	position vector
s	temperature-independent source term
t	time
T	temperature
u	ambient temperature
κ	thermal conductivity

ξ coefficient of temperature-dependent source term

ν_{ij}^{m} transition probability

Ω solution domain

Γ_1 boundary of the first kind

Γ_2 boundary of the second kind

Γ_3 boundary of the third kind

Subscript and superscripts

i, j, k space indices

l, m time indices

7. Acknowledgment

The author would like to acknowledge the financial support from the National Science and Technology Development Agency.

8. References

Emery, A. F. and Carson, W. W., 1966, "A Modification to the Monte Carlo Method — The Exodus Method," *ASME Journal of Heat Transfer*, Vol. 90, pp. 328-332.

Haji-Sheikh, A. and Sparrow, E. M., 1965, "The Solution of Heat Conduction Problems by Probability Methods," *ASME Journal of Heat Transfer*, Vol. 89, pp. 121-131.

Naraghi, M. H. N. and Tsai, S. C., 1993, "A Boundary-Dispatch Monte Carlo (Exodus) Method for Analysis of Conductive Heat Transfer Problems," *Numerical Heat Transfer, Part B*, Vol. 24, pp. 475-487.

Sadiku, M. N. O. and Hunt, D. T., 1992, "Solution of Dirichlet Problems by the Exodus Method," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 40, pp. 89-95.

Sadiku, M. N. O., Ajose, S. O., and Fu, Z. B., 1994, "Applying the Exodus Method to Solve Poisson Equation," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 42, pp. 661-666.

Zinsmeister, G. E. and Sawyerr, J. A., 1974, "A Method for Improving the Efficiency of Monte Carlo Calculation of Heat Conduction Problems," *ASME Journal of Heat Transfer*, Vol. 96, pp. 246-248.