Identification of buckling load of thin plate using the vibration correlation technique

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Abstract

Stability is one of the important failure modes of thin-walled structures subjected to compressive loading. Besides theoretical and numerical studies, buckling of plate problem has been experimentally investigated. In this paper, the vibration correlation technique (VCT) is introduced as an alternative method to determine the buckling load. The relationship between applied in-plane load and the natural frequency of plates are derived from the differential governing equations of both problems. In this technique, square of the natural frequency of flexural vibration of plate is plotted against the applied in-plane load. It is shown that when the applied load approaches the buckling load of plate, the natural frequency of plate approaches zero. The square of the natural frequency is also linearly related to the applied load. Thus, the buckling load can be determined by extrapolating the data to the applied load at which natural frequency approaches zero. The vibration correlation technique is numerically verified by plotting square of natural frequency of loaded plate with applied in-plane load. The obtained buckling load from the plot is successfully compared with the buckling load determined by direct numerical method. The Ritz method along with the beam functions is employed to determine the natural frequency and the buckling load of rectangular isotropic plate with combined boundary conditions. Besides buckling load, buckling mode can also be determined from vibration mode. The specimens used in this study are rectangular isotropic plates with simple-clamped-simple-clamped (S-C-S-C) and simple-clamped-simple-free (S-C-S-F) boundary conditions.

Keywords: Buckling, Vibration, Plate, Vibration correlation technique, Ritz method.

1. Introduction

Stability is one of the important factors that should be considered in design of thin-walled structures subjected to compressive loading. Besides buckling of columns and shells, buckling of plates is a problem that has been in the interest of many structural engineers and researchers. Studies in this field include theoretical, numerical, and experimental investigations. Identification of the buckling point of isotropic rectangular plates with simple support on all edges has been studied by Supasak [1]. In that study, buckling loads of aluminum plates were identified from the experiment using four different methods; i.e. 1) a plot of in-plane loads vs. out-of-plane displacement, 2) a plot of in-plane loads vs. end-shortening, 3) a plot of in-plane loads vs. difference of surface strains, and 4) a plot of the ratio of out-of-plane displacement to in-plane load vs. out-of-plane displacement. Experimental buckling loads determined from the first three methods have a fairly high percent error compared with the theoretical solutions. The last identification method gave the value of buckling loads with percent error as high as 69% compared with the theoretical solutions. The author also indicated the difficulty in identifying the buckling load from the plots of measured data. Chai et. al. [2] compared experimental buckling load of composite plates with the theoretical solutions. The discrepancy between the experimental and theoretical solutions was ranged between -7 % and +11 %. Tuttle et. al. [3] determined buckling loads from plots of applied in-plane load vs. out-of-plane displacement of composite panels and compared the experiment results to numerical predictions obtained from Galerkin method. Although the average percent error between the measured and predicted buckling loads is very low, the standard deviation of the percent error is as high as 15%. This high deviation reflects the accuracy of the measurements. Thus, it is difficult to experimentally determine the buckling load of plates using static test method, since even the smallest amount of imperfection of the specimen, loading apparatus, or boundary conditions can have an apparent impact on the buckling behavior. Moreover, in the static approach, there is a need to draw two lines in the pre-buckling and post-buckling regions which may be a cause of error.

There is a need for an alternative approach to experimentally identify the buckling load of plate. In this paper, the vibration correlation technique (VCT) which is a dynamic approach is explored. Lurie and Monica [4] shown that square of the frequency of the lateral vibration of thin plate with simple supports on all edges is linearly related to the end load. They also conducted some experiments on elastically restrained columns, rigid-joint trusses, and thin flat plates. The authors reported that VCT was successfully employed to predict buckling load of only columns and truss. For flat plates, because of the initial curvature, the buckling load cannot be predicted by the proposed method. However, Chailleux et. al. [5] showed later that with a careful experiment setting, VCT can be used to determine the buckling load with satisfied accuracy.

In this research, the relationship between buckling and vibration behavior of thin plate is investigated. The relationship between applied in-plane load and the natural frequency of plates are derived from the differential governing equations of both problems. The derived relationship is verified using a numerical method. This relationship also implies that buckling load of plate can be obtained from the vibration data of the loaded plates. So, an alternative method for buckling load identification using dynamic approach is proposed.

2. Relationship between vibration and buckling behaviors

In this study, the vibration and buckling behaviors of a rectangular isotropic plate as shown in Fig.1 are investigated. The buckling load of plate represented by $\bar{N}_x$ is the in-plane load N_x at which buckling occurs. For vibration behavior, the natural frequencies of plate can be determined for a specimen with a given tensile or compressive load N_x.

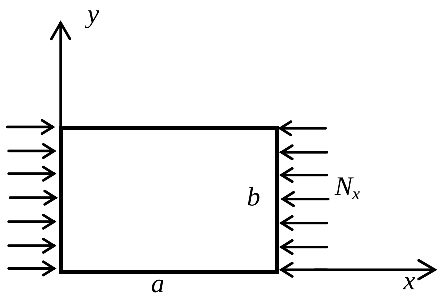


Figure. 1 A rectangular plate subjected to a uniaxial in-plane load

The governing equation for buckling and vibration of thin isotropic plate can be written as;

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{\bar{N}_x}{D}\frac{\partial^2 w}{\partial x^2} = 0 \qquad (1)$$

and

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} - \frac{\omega^{*2}\rho}{D} w = 0, \qquad (2)$$

respectively.

where w = out-of-plane displacement

ρ = mass of plate per unit area

$D = \frac{Et^3}{12(1-\nu^2)}$ (Plate flexural rigidity)

$\bar{N}_x$ = buckling load

N_x = applied in-plane load

ω^* = natural frequency of the plate with applied in-plane load N_x

It should be noted that $\bar{N}_x$ and N_x refer to the same in-plane load, however, $\bar{N}_x$ is the buckling load which must be a compressive load (negative value), while N_x is the applied in-plane load which can be either tension or compression.

For a given rectangular plate, the relationship between the natural frequency and an applied in-plane load N_x can be determined by considering the governing equations, Eq.(1 and 2). For a specimen with a given boundary conditions, it is widely known that buckling mode and vibration mode of the plates are identical. Specifically, the out-of-plane displacement of the buckled plate is identical to the out-of-plane displacement of one of the vibration mode. So, for a given specimen, the governing of the buckling problem can be rewritten as.

$$L_1(w) - \bar{N}_x L_2(w) = 0 \qquad (3)$$

where $L_1(w) = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$

$L_2(w) = \frac{1}{D}\frac{\partial^2 w}{\partial x^2}$

Similarly, the governing of the vibration of loaded plates is written as;

$$L_1(w) - N_x L_2(w) - \omega^{*2} L_3(w) = 0 \qquad (4)$$

where $L_3(w) = \frac{\rho w}{D}$

It should be noted that the terms contained derivatives of w for both problems are the same because the buckling mode and vibration mode are identical. From Eq.(3), the buckling load of plate can be written as;

$$\bar{N}_x = \frac{L_1(w)}{L_2(w)} \qquad (5)$$

Similarly, the natural frequency of plate with and without the applied in-plane load can be written as;

$$\omega^{*2} = \frac{L_1(w) - N_x L_2(w)}{L_3(w)} \qquad (6)$$

and $$\omega^2 = \frac{L_1(w)}{L_3(w)} \qquad (7)$$

where

ω^* is natural frequency of a plate with applied load N_x

ω is natural frequency of a plate without applied load

From Eq.(5-7), ratio of the square of natural frequency of the loaded plate to that of the unloaded plate is written as;

$$\left(\frac{\omega^*}{\omega}\right)^2 = 1 - \frac{N_x}{\bar{N}_x} \qquad (8)$$

Since buckling load $\bar{N}_x$ and natural frequency of the unloaded plate ω is constant for a given specimen, it is concluded that square of the natural frequency of the loaded plate ω^{*2} is linearly varied with the applied load N_x. Since this relationship is derived from the governing, it is independent of boundary conditions.

From the linear relationship between ω^{*2} and N_x shown in Eq.(8), with the buckling load being a negative value, it is notice that the natural frequency of the plate increases with the applied tensile load. On the other hand, it is decreased with the applied compression. Moreover, if the applied load N_x equals the buckling load of the plate, the natural frequency ω^* theoretically equals zero. With this observation, ones can utilize the natural frequencies of the loaded plate to predict the buckling load of plate by plotting ω^{*2} versus the in-plane load N_x. The buckling load could be determined from the applied load N_x at which the natural frequency approaches zero.

3. Numerical investigation

To verify the relationship between the natural frequency and buckling load of plate, natural frequencies of the loaded plate and buckling load of plate are determined. The vibration mode and buckling load are also investigated. Since the closed form solutions are available for all edges simple support (SSSS) specimen only, the numerical method is used in this study. Both vibration and buckling problems are solved using the Ritz method [6]. The total potential energy for the vibration of loaded plate can be written as;

$$\Pi = \frac{1}{2}\iint \left[D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)D\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right) + N_x\left(\frac{\partial w}{\partial x}\right)^2 - \rho\omega^{*2}w^2 \right] dxdy \quad (9)$$

and the total energy for the buckling problem is represent by;

$$\Pi = \frac{1}{2}\iint \left[D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)D\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right) + \bar{N}_x\left(\frac{\partial w}{\partial x}\right)^2 \right] dxdy \quad (10)$$

To determine the natural frequency of the loaded plate, Eq.(9) is considered by treating N_x as a applied load which is known and ω^* is the unknown to be determined. For buckling behavior, the total energy in Eq.(10) is used with an unknown variables $\bar{N}_x$. To solve both problems, the out-of-plane displacement w is assumed to be;

$$w(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn} X_m(x) Y_n(y) \quad (11)$$

A_{mn} are the unknown coefficients representing vibration mode or buckling mode. $X_m(x)$ and $Y_n(y)$ are the basis functions satisfied the boundary conditions at $x = 0$, $x = a$ and $y = 0$, $y = b$, respectively. In this study, beam function is chosen as basis functions. For simple support on both ends, the function is represented by a well-known double sine series. For other boundary conditions, the beam functions can be written in form of; [7]

$$\varphi_m(r) = \gamma_m \cos\left(\frac{\lambda_m r}{L}\right) - \gamma_m \cosh\left(\frac{\lambda_m r}{L}\right) + \sin\left(\frac{\lambda_m r}{L}\right) - \sinh\left(\frac{\lambda_m r}{L}\right) \quad (12)$$

where φ is either X or Y, and r can be x or y. The values of γ_m and λ_m depend on the boundary condition of the plate. For case of clamp boundary condition on both ends, λ_m can be determined from roots of;

$$\cos\lambda_m \cosh\lambda_m = 1,$$

and γ_m is determined from;

$$\gamma_m = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m}.$$

For clamp-free boundary condition where one end is clamped and one end has no support, λ_m is determined from roots of;

$$\cos\lambda_m \cosh\lambda_m = -1$$

and γ_m is determined from;

$$\gamma_m = \frac{\cos\lambda_m + \cosh\lambda_m}{\sin\lambda_m - \sinh\lambda_m}.$$

The basis functions for the first four modes for the cases of clamp-clamp boundary condition and clamp-free boundary conditions are plotted as examples in Fig.(1)

To solve for the natural frequency and buckling load, the total potential energy is determined for each problem by substituting the approximate displacement functions Eq.(11) into the total potential energy Eq.(9,10), respectively. The displacement functions must be selected according to the boundary conditions of the plate. After performing integrations, the total potential energy is written in term of the undetermined coefficients A_{mn} and the natural frequency ω^* or buckling load $\bar{N}_x$ for vibration and buckling problems, respectively. According to the principle of minimum total potential energy, the total potential energy is minimized with respect to the unknown coefficients A_{mn} according to;

$$\frac{\partial \Pi}{\partial A_{mn}} = 0 \quad (13)$$

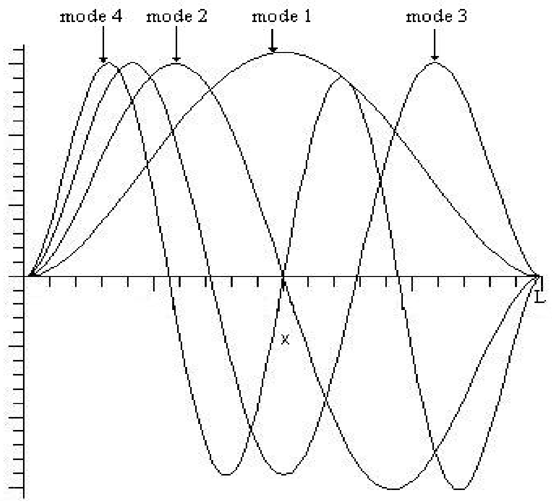
Eq.(13) is a system of $M\times N$ linear equations, which can be rearranged as a matrix form of generalized eigenvalue problems as:

$[A][C] - \omega^{*2}[B][C] = 0$, for vibration problem, and (14a)

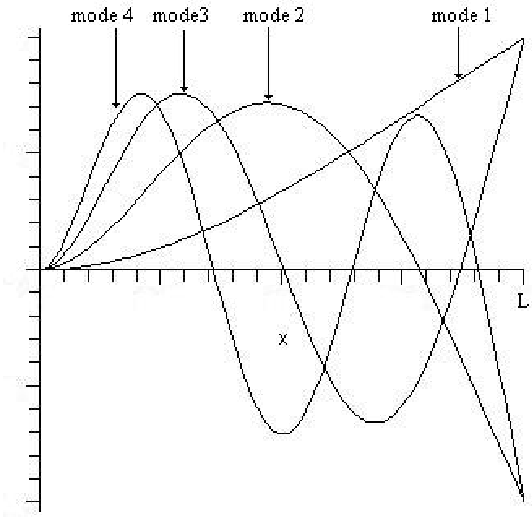
$[A][C] + \bar{N}_x[B][C] = 0$, for buckling problem. (14b)

where $[A]$ and $[B]$ are square matrices whose elements are determined from the plate properties. [C] is a column matrix of eigenvector A_{mn}. ω^{*2} and $\bar{N}_x$ are the eigenvalues representing square of natural frequency and buckling load of plate, respectively. A number of eigenvalues will be obtained after the generalized eigenvalue problem equation, Eq.(14), is solved. For vibration problem, each eigenvalue is square of the natural frequency of plates. However, only the lowest eigenvalue of Eq.(14b) is the buckling load which is of interest in buckling problem. The corresponding eigenvectors of each is used to determine the vibration mode or buckling mode by substituting into the displacement function Eq.(11).

Before implementing the Ritz method, convergence studies was performed to ensure that the number of term used in the displacement function is enough to give a converged solution. An aluminum rectangular plate is used in the convergence study. The mechanical properties of aluminum are assumed to be E = 70 GPa, v = 0.3, and ρ = 2707 kg/m^3 with plate thickness of 2 mm. The convergence of a rectangular plates with $a = b = 200$ mm, and all edge clamp boundary condition is shown in Fig 2. It is observed that the buckling load converges when the value of m and n in the displacement function equals 5. The value of m and n used in this study is 12, i.e. there are 144 terms in the displacement function.



(a)



(b)

Figure 1. Displacement functions for (a) clamp-clamp boundary condition and (b) clamp-free boundary conditions.

4. Numerical results

In this study, three cases of aluminum plates are investigated using a numerical method outlined in the previous section. Dimensions, boundary conditions, and theoretical bucking load of plate are summarized in Table 1. The buckling loads are determined from the solution of generalized eigenvalue problem, Eq.(14b). This buckling load is considered herein as the "theoretical solution." Specimens with two different combinations of boundary condition are investigated. For SCSC boundary condition, the first letter S and third letter S represent the boundary condition on the $x = 0$ and $x = a$ edges, respectively. Similarly, the second and fourth letters represent the boundary condition on the $y = 0$, and $y = b$ edges, respectively. To verify the relationship shown in Eq.(8), the natural frequency of the loaded plate is determined for different applied in-plane loads N_x. The in-plane load can be either tension, compression, or no load. A generalized eigenvalue problem shown in Eq(14a) is set for the specimens with a particular applied load N_x. This applied load is treated as a known and constant value. The obtained square of natural frequency for each vibration mode is plotted against the applied in-plane load as shown in Fig. 3-5, for all specimens, respectively.

Table 1. Dimensions and boundary conditions of the specimen

Specimen No.	Dimension $a \times b$ (mm^2)	Boundary Condition	Buckling load $\bar{N}_x$, (kN/m)
1	200×200	SCSC	-97.323
2	400×200	SCSC	-88.215
3	200×200	SCSF	-21.019

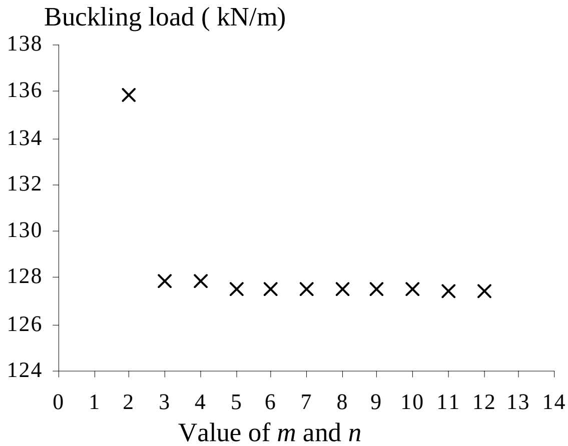


Figure 2. Convergence of the buckling load of CCCC aluminum plates

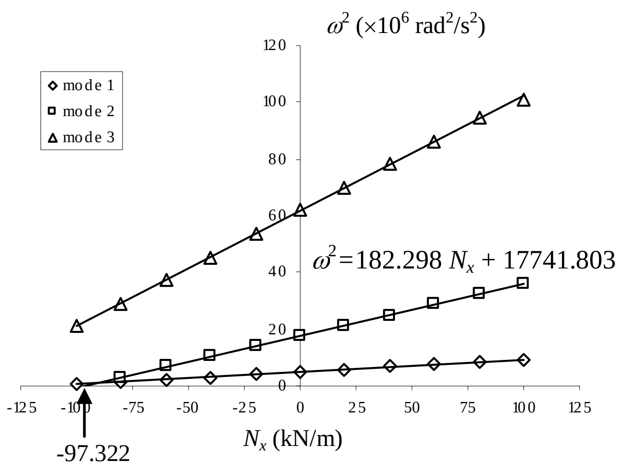


Figure 3. Plot of ω^{*2} and N_x of specimen No. 1

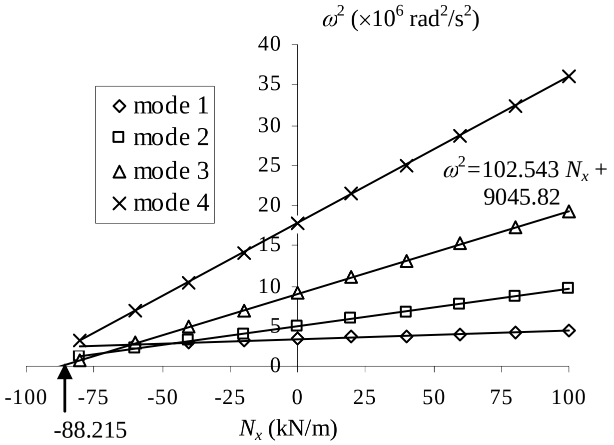


Figure 4. Plot of ω^{*2} and N_x of specimen No. 2

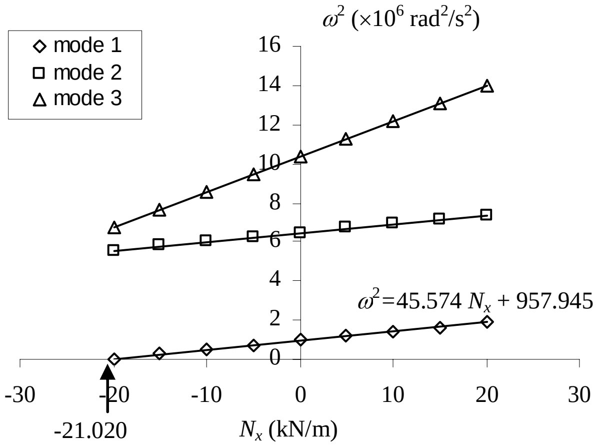
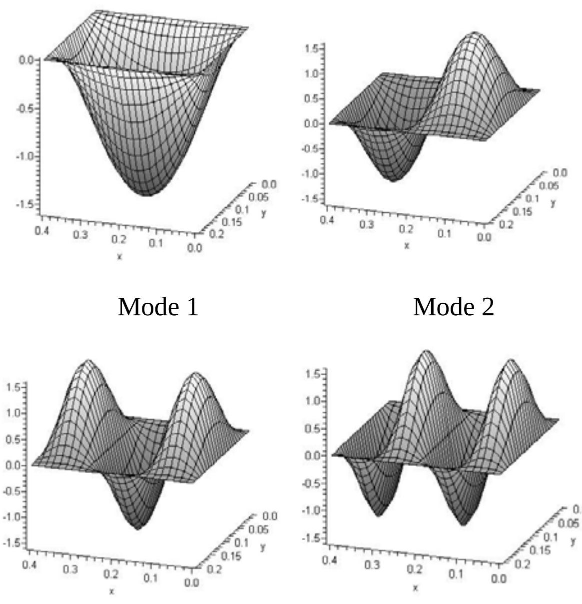


Figure 5. Plot of ω^{*2} and N_x of specimen No. 3

From Fig 3-5, the relationship between ω^{*2} and N_x is linear as expected. It is also shown that the natural frequency is increased as the in-plane load becomes higher in the tensile direction (positive N_x). On the other hand, the natural frequency approaches zero when the applied load is amplified in the compressive direction (negative N_x). The plots of ω^{*2} vs. N_x can be used to verified the relationship shown in Eq.(8) by extrapolating the value of N_x at which the natural frequency becomes zero. The extrapolation can be systematically performed by determined the equation representing the relationship between ω^{*2} and N_x for each mode of vibration and solved for N_x for zero natural frequency. The obtained N_x at zero natural frequency of each vibration mode are compared with each other. The lowest value of N_x at zero natural frequency shown in the figures is the predicted buckling load. In the figure, only the equation of ω^{*2} and N_x of the vibration mode with the lowest value of N_x at zero natural frequency is presented.

For the SCSC specimens with aspect ratios of 1 and 2, it is found that the predicted buckling loads are -97.322 and -88.215 kN/m, respectively, which are practically identical to the theoretical ones. The buckling mode for specimen No. 1 is mode 2 since the plot of mode 2 vibration intersects the applied load axis before other modes. With different aspect ratio, the predicted buckling mode for specimen No. 2 is mode 3. Predicted buckling modes for both cases are agreed with the solutions obtained from the buckling problem. Fig.6 shows the vibration mode corresponding to the vibration data shown in Fig.4. The theoretical buckling mode for specimen No.2 which is mode 3 is presents in Fig.7. Clearly, the predicted bucking mode using vibration data matches the theoretical solution very well. Besides specimen with SCSC boundary conditions, a SCSF specimen is also investigated. It is found that the derived relationship between ω^{*2} and N_x can be used to predicted the buckling load with very good accuracy. The predicted buckling load for the case of SCSF specimen is 21.020 kN/m compared with the theoretical solution of 21.019 kN/m. The buckling mode is also very well predicted.



Mode 3 Mode 4

Figure 6. Vibration mode shapes of specimen No.2

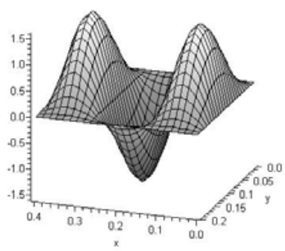


Figure 7. Buckling mode of specimen No.2

5. Conclusion

This research investigates the vibration response of isotropic rectangular plates subjected to uniform in-plane load. By considering the governing equations of the vibration and buckling problems, it is shown that square of the natural frequency of the loaded plate is linearly varied with the applied load. The natural frequency is increased with the tensile load and decrease with the compressive load. This relationship is determined without a need to solve the differential governing equations, so it is applicable for plates with any boundary conditions. It is also shown that the square of the nature frequency approaches zero when the in-plane load approaches the buckling load. The derived relationship is verified by theoretically solving the vibration and buckling problems of specimens with combinations of boundary conditions. The Ritz method is employed to determine natural frequency of the loaded plate and buckling load of plate. In the process, vibration mode shape and buckling mode are also obtained. From the study, the predicted buckling load and mode from the vibration data are corresponded to the theoretical solution very well.

The derived relationship between square of the natural frequency and the applied load can be used as an alternative method of identifying the buckling load experimentally. The advantage of dynamic approach over the static approach is that, in the dynamic approach, there is no need to drawn lines in the pre-buckling and post-buckling region. So, the error from human judgment can be eliminated. For future study, the derived relationship should be verified with the measurement data.

Acknowledgments

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References

[1] Supasak, C., 2005. Conparison of buckling loads of thin plates by experiment method. Master Thesis, Mechanical Engineering Department, Chulalongkorn University, Bangkok, Thailand.

[2] Chai, G.B., Banks, W.M., and Rhodes, J., 1991. An experiment study on laminated panels in compression. Composite Structures, Vol. 19, No. 1, pp. 67-87.

[3] Tuttle, M., Singhatanadgid, P., and Hinds, G., 1999.3 Buckling of composite panels subjected to biaxial loading. Experiment Mechanics, Vol. 39, No. 3, pp. 191-201

[4] Lurie, H. and Monica, S., 1952. Lateral vibrations as related to structural stability. Journal of Applied Mechanics, pp. 195-204.

[5] Chailleux, A., Hans, Y., and Verchery, G. 1975. Experimental study of the buckling of laminated composite columns and plates. Journal of Mechanical Sciences, Vol. 17, pp. 489-498.

[6] Pannok, C. and Singhatanadgid, P., 2006 Buckling analysis of composite laminate skew plates with various edge support conditions Proceedings of the 20th Conference of the Mechanical Engineering Network of Thailand (ME-NETT 20), Nakhon Ratchasima, Thailand. 18-20 October 2006.

[7] Weaver Jr., W., Timoshenko, S.P., and Young, D.H. 1990. Vibration problems in engineering. Wiley, New York, USA.