An LMI-Based Output Feedback Controller Design for Control of A Spring Connected Double Inverted Pendulum

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Abstract

A linear spring connected double inverted pendulum as purposed, which is nonlinear and unstable as purposed by Hou et al. [1, 2] and Hongxing [5]. Then, the nonlinear system is linearized about the equilibrium point in order to simplify the system which allows technique in linear control to stabilize the system. The purpose of this paper is to show that an LMI-based output feedback method can be used to stabilize to system when all state variables are not available. The LMI-based output feedback method is applied to solve an example of linear spring connected double inverted pendulum and compared with LQR method. Finally, the simulation results of both methods are shown and discussed.

***Keywords**:* LMI-based output feedback, a linear spring connected double inverted pendulum, optimal control, and LQR technique

1. Introduction

A double inverted pendulum system, which is nonlinear and unstable, is modified by connecting the mass carrying the double inverted pendulum with an additional by a spring. The new system is a linear spring connected double inverted pendulum as purposed by Hou et al. [1, 2] and Hongxing [5]. A double inverted pendulum system which is nonlinear and unstable is modified by connecting the mass carrying the double inverted pendulum with an additional by a spring. Therefore, according to this modification, the original double inverted pendulum system becomes a more complicate and challenging problem in nonlinear control and stabilization area.Fortunately, the system can be approximated and simplified as a linear time invariant system by using linearization about the equilibrium point. Therefore, many techniques in linear full state feedback control can be applied to stabilized the system such as PID control, LQR etc. instead of using nonlinear control which is more complicate. However, in practical all state variables are immeasurable or in some situations the designer intend to reduce the number of measurement signals for an economic reason. The popular method in controller design such as an LMI-based output feedback is the appropriate method in order to stabilize the linearized system.

This paper is organized as follows. First, the mathematical representing the linear spring connected inverted pendulum is stated. Second, the concepts of the LMI-based output feedback is purposed. Next, the simulation results of both LQR and the LMI-based output feedback are presented. Finally, discussion and conclusions are stated.

2. A Linear Spring Connected Double Inverted Pendulum System

2.1 Mathematical Model

A linear spring connected double inverted pendulum as purposed by Hou et al.[1, 2] and Hongxing[5]. as shown in figure1. The mathematical model representing the equation of motion containing 4 degree of freedom can be derived by using Lagrange equation or Newton's Law as shown in Eq. (1) ,Hou et al.[1, 2].

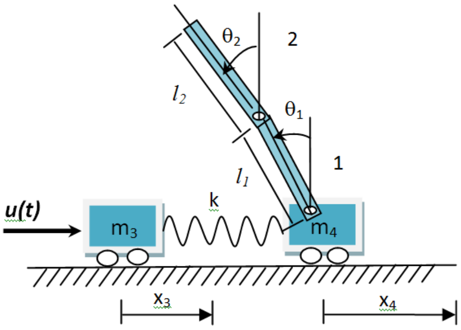

Fig. 1 A Linear spring Connected Double Inverted Pendulum System

$$
\begin{aligned}
&-m_2 l_{C2}\cos\theta_2 \ddot{x}_4 + m_2 l_1 l_{C2}\cos(\theta_1-\theta_2)\ddot{\theta}_1 \\
&+(J_{C2}+m_2 l_{C2}^2)\ddot{\theta}_2 = m_2 g l_{C2}\sin\theta_2 + \sin\theta_2 \\
&+m_2 l_1 l_{C2}\sin(\theta_1-\theta_2)\dot{\theta}_1^2 - c_2(\dot{\theta}_1-\dot{\theta}_2)
\end{aligned}
$$

$$
\begin{aligned}
&-(m_1 l_{C1}+m_2 l_1)\cos\theta_1 \ddot{x}_4 + (J_{C1}+m_1 l_{C1}^2 + m_2 {l_1}^2)\ddot{\theta}_1 \\
&+m_2 l_1 l_{C2}\cos(\theta_1-\theta_2)\ddot{\theta}_2 = m_1 g l_{C1}\sin\theta_1 + m_2 g l_1 \sin\theta_1 \\
&-m_2 l_1 l_{C2}\sin(\theta_1-\theta_2)\dot{\theta}_2^2 - c_1\dot{\theta}_1 - c_2(\dot{\theta}_1-\dot{\theta}_2)
\end{aligned}
$$

$$
\begin{aligned}
&(m_4+m_1+m_2)\ddot{x}_4 - (m_1 l_{C1}+m_2 l_1)(\mu\sin\theta_1+\cos\theta_1)\ddot{\theta}_1 \\
&-(\mu m_2 l_{C2}\sin\theta_2 + m_2 l_{C2}\cos\theta_2)\ddot{\theta}_2 \\
&= \mu(m_1 l_{C1}+m_2 l_1)\dot{\theta}_1^2\cos\theta_1 + \mu m_2 l_{C2}\dot{\theta}_2^2\cos\theta_2 \\
&-(m_1 l_{C1}+m_2 l_1)\dot{\theta}_1^2\sin\theta_1 - m_2 l_{C2}\dot{\theta}_2^2\sin\theta_2 - k(x_4-x_3) \\
&-\mu(m_4+m_1+m_2)g - c\dot{x}_4 \\
&m_3\ddot{x}_3 = F(t)+k(x_4-x_3)-\mu m_3 g - c\dot{x}_3 \qquad (1)
\end{aligned}
$$

where m_3 = mass of cart 3, m_4 = mass of cart 4, c_0 = friction factor, c = tack coefficient, and $u(t)$ = control input.

2.2 Problem Statement

By Letting $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$, $z_4 = \theta_2$, $z_5 = \dot{x}_3$, $z_6 = \dot{x}_4$, $z_7 = \dot{\theta}_1$, and $z_8 = \dot{\theta}_2$, the equation of motion in Eq. (1) can be converted in the form of a first order differential equation , $\dot{\underline{z}} = f(\underline{z}(t), u(t))$ where $\underline{z}(t) = [z_1, z_2, z_3, z_4, z_5, z_6]^T$. Then, the linear system can be expressed in the state space form as

$$\dot{\underline{z}} = A\underline{z} + Bu \text{ and } y = C\underline{z} + Du \qquad (2)$$

where $u(t)$ = control input,
$y(t)$ = output variable,
A = State matrix,
B = input matrix,
C = Output matrix and
D = input-to-output coupling matrix.

3. LMI-Based Output Feedback Control

In the situation that all state variables cannot be measurable or designer intend to reduce the number of output variable, LMI-Based Output Feedback Controller design is applied to stabilize the linear spring connected double inverted pendulum expressed in Eq. (1).

A linear matrix inequality (LMI) is regarded as a convex constraint. Consequently, optimization problems with convex objective functions and LMI constraints are solvable relatively efficiently with off-the-shelf software, i.e., MATLAB LMI Control Toolbox. The form of an LMI is very general. Linear inequalities, convex quadratic inequalities, matrix norm inequalities, and various constraints from control theory such as Lyapunov and Riccati inequalities can all be written as LMIs. Further, multiple LMIs can always be written as a single LMI of larger dimension. Thus, LMIs are a useful tool for solving a wide variety of optimization and control problems. In this paper, an output feedback controller design based on [3]-[4] can be constructed in terms of LMIs constraints.

3.1Output Feedback Controller Design

The linearized systems are derived in the previous section and designed to achieve the performance requirements given in the Problem Statement by using a full order output feedback controller. We use an LMI-based controller design methodology to achieve the following.

Theorem For the linearized system, the closed-loop system can accomplish the expected performance requirements: All closed-loop poles are located in the open left-half plane if and only if there exist symmetric matrices X, Y and matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ such that the following LMIs simultaneously satisfied.

$$\begin{pmatrix} \Omega_{11} & \Omega_{21}^T \\ \Omega_{21} & \Omega_{22} \end{pmatrix} < 0, \qquad \begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0$$

where

$$\Omega_{11} = AX + XA^T + B_u\bar{C} + (B_u\bar{C})^T,$$

$$\Omega_{21} = \bar{A} + (A + B_u\bar{D}C_y)^T,$$

$$\Omega_{22} = A^TY + YA + \bar{B}C_y + (\bar{B}C_y)^T$$

for X and Y symmetric matrices, $X > Y$ means $X - Y$ is positive definite.

A dynamic output feedback controller can be constructed as follows:

$$A_K := (N^T)^{-1}(\bar{A} - NB_KC_yX - YB_uC_KM^T - Y(A + B_uD_KC_y)X)(M^T)^{-1},$$

$$B_K := (N^T)^{-1}(\bar{B} - YB_uD_K),$$

$$C_K := (\bar{C} - D_KC_yX)(M^T)^{-1}$$

$$D_K := \bar{D}$$

where X and Y are arbitrary nonsingular matrices satisfying $MN^T = I - XY$ .The details of proof is purposed by Chilali and Gahinet [3].

4. Simulation Results and Discussion

In order to show the feasibility of the LMI-based output feedback method. It is applied to solve an example of linear spring connected double inverted pendulum and compared with LQR method. The implementation is done in MATLAB, and the simulation results of both methods are shown and discussed.

The example of a linear spring connected double inverted pendulum system is defined by letting the parameters of the system as follows: $m_1 = 0.25$ kg, $m_2 = 0.25$ kg, $m_3 = 1.5$ kg, $m_4 = 1.5$ kg, $c_0 = 0$, $c = 0$, $l_1 = 0.4$ m, $J_{c1} = 0.0033$ kg·m², $l_{C1} = 0.2$ m, $c_1 = 0.05$, $l_2 = 0.4$ m, $J_{c2} = 0.0033$ kg·m², $l_{C2} = 0.2$ m, $c_2 = 0.05$, $k = 100$ N/m, $g = 9.81$ m/s^2 in appropriate SI units.

The simulation results given by both LMI based output feedback and LQR methods which are time response of measureable state variables, $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_3$, and $z_4 = \theta_4$ are shown in Fig. 4 in blue curves and red dash curves respectively. The poles of the close loop control system of LMI based output feedback method located on the left half plane as shown in Fig. 3. Using control input signal of the LMI-based output feedback method as shown in Fig. 3 can stabilize the system. All measureable state variables converge to equilibrium point at zero as time increased to steady state as shown in Fig. 3 . Therefore, it is shown that the LMI-based output feedback can stabilize the system at the equilibrium point.

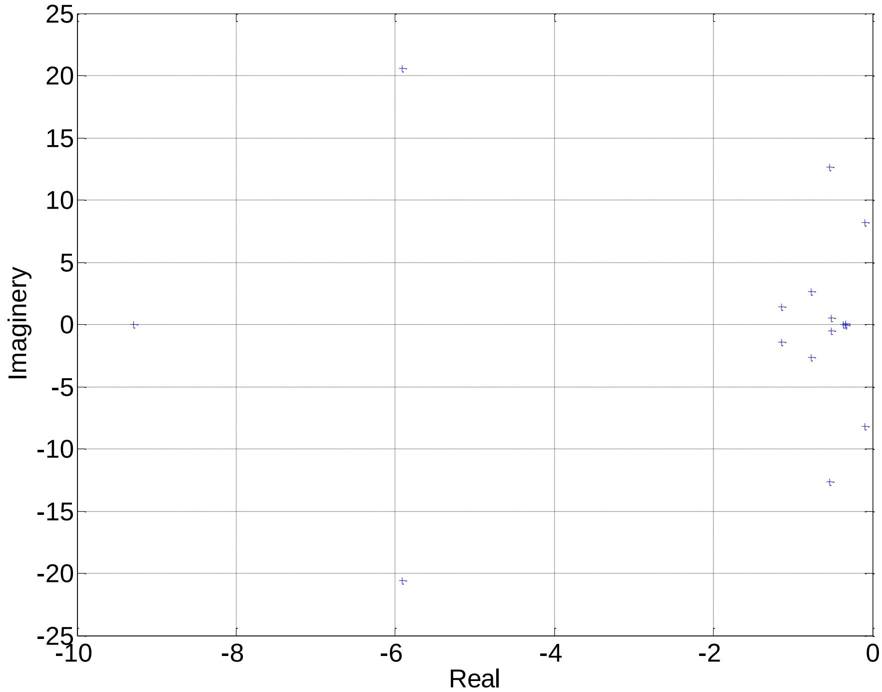

Fig. 2 Poles of the close loop control system of LMI-based output feedback method

Considering the simulation given LQR method , the LQR method can stabilized as the values of state variables, $z_3 = \theta_3$ and $z_4 = \theta_4$ converge to zero . However, the carts are not at the equilibrium point at the steady state as the state variables $z_1 = x_3$ and $z_2 = x_4$ approach to finite values. Considering the simulation results given by LMI output feedback method, it is clear that LMI based output feedback method can stabilize the system as the all values of state variables, $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_3$, and $z_4 = \theta_4$ converge to zero at the steady state.

Also for economically reasons, the LMI-based output method requires lower number of measurement signals than the LQR method does.

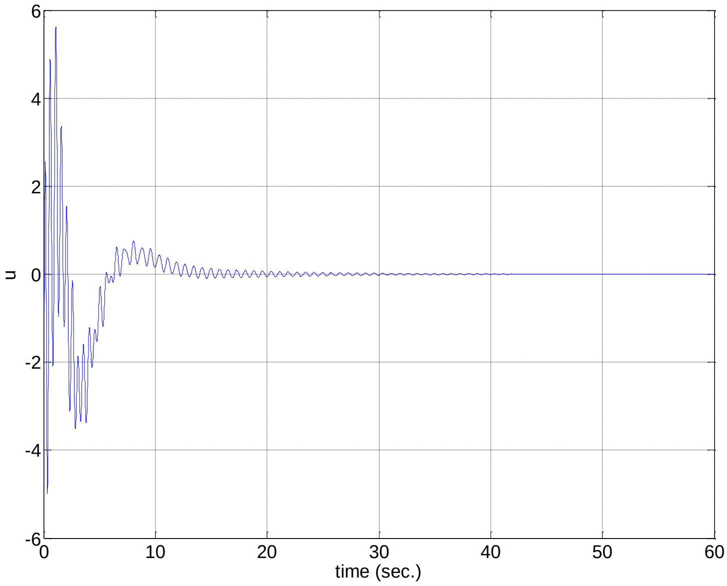

Fig. 3 A control input signal of the LMI-based output feedback method

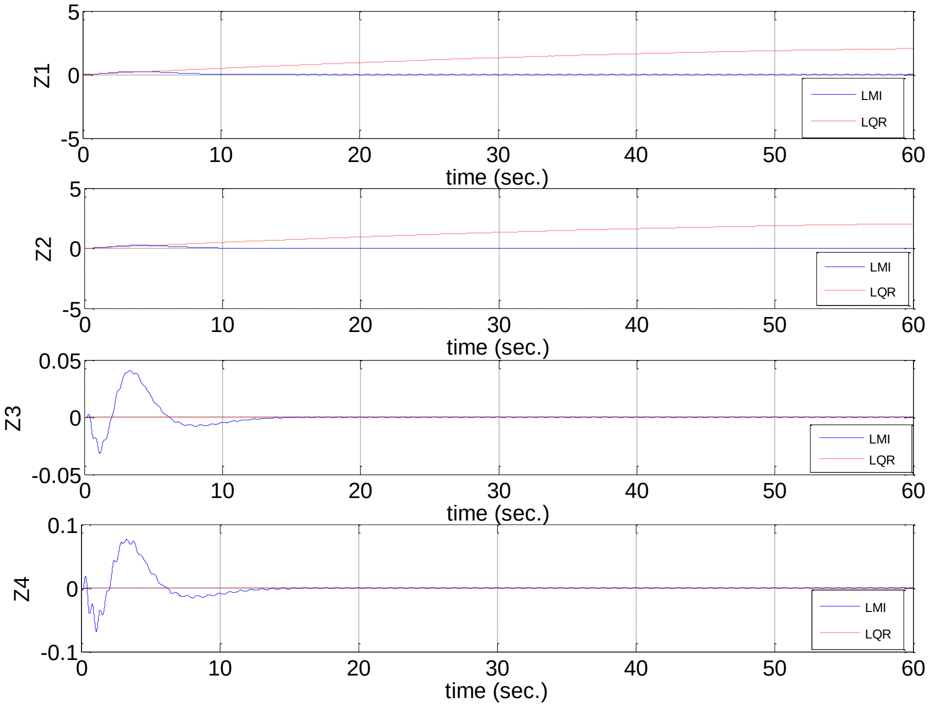

Fig. 4 Time responses of measureable state Variables of Z_1 ,Z_2, ,Z_3 ,and Z_4 of the LMI-based output feedback method(blue curves) and LQR method(red dash curves).

5.Conclusions

Considering the simulation results, it can be conclude as follows. First, the LMI-based output feedback is feasible to stabilize the mass spring connected double inverted pendulum system. Second, the LMI-based output method performs better and more efficiently than the LQR does, since the LMI-based output feedback drives state variables that can be measured corresponding to converge to zero at steady state and requires lower number of measurement signals. Therefore, it is appropriate to apply the LMI-based output feedback controller to a linear spring connected double inverted pendulum system for both efficiency and economical reason.

6. References

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