H-infinity Controller Design for A Linear Spring Connected Double Inverted Pendulum

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Abstract

A modified double inverted pendulum known as a linear spring connected double inverted pendulum is highly nonlinear and unstable as purposed by Hou et al. [1,2] and Hongxing [7]. However, the system can be simplified to a linear time invariant system through linearization about a pre-specified equilibrium point. In practice, the system is unavoidably affected by an exogenous disturbance, the robust control technique is an appropriate method to deal with this situation. The popular robust H infinity can be also applied to solve this problem if the energy of the disturbance is bounded. Then, the H infinity controller is applied to an example of the linear spring connected double pendulum compared with LQR method through simulation.

Keywords: H infinity control, a linear spring connected double inverted pendulum ,optimal control, robust control, and LQR technique.

1. Introduction

A double inverted pendulum becomes a more challenging control problem by connecting another mass with the mass carrying a double inverted pendulum through a linear spring. The modified system is a linear spring connected double inverted pendulum system. As general double inverted pendulum, this systems, a linear spring connected double inverted pendulum system is unstable and highly nonlinear as purpose in [1-2] and [7]. Therefore, this challenging problem can be used as a test problem for control techniques. Instead of using nonlinear control techniques to solve this problem, a linear spring connected can be approximated as a linear time invariant through linearization about the pre-specified equilibrium point; upright position of the double inverted pendulum. Therefore, many techniques in linear control methods can be applied to this problem. However, in reality, the system is subjected to disturbance. In the situation when the energy of disturbance is bounded, H infinity controller is appropriate controller to stabilize a linear time invariant system under disturbance. This method was used by Tsachouridis et al.[4] to stabilize the triple inverted pendulum system.

2. A Linear Spring Connected Double Inverted Pendulum System

2.1 Mathematical Model

A linear spring connected double inverted pendulum is constructed from a general double inverted pendulum as shown in Fig. 1 The mathematical model of a linear spring connected double inverted pendulum system can be derived and shown in Eq.(1) as purposed by Hou et al. [1,2] .

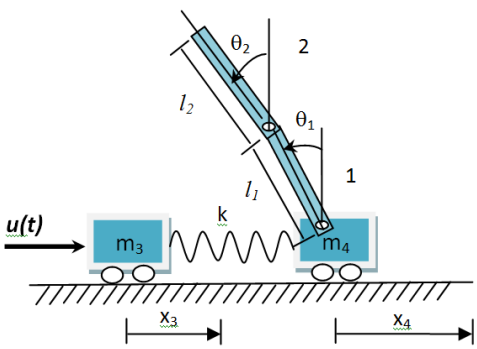

Fig. 1 A linear spring connected double inverted pendulum system

$$-m_2 l_{C2}\cos\theta_2 \ddot{x}_4 + m_2 l_1 l_{C2}\cos(\theta_1-\theta_2)\ddot{\theta}_1 + (J_{C2}+m_2 l_{C2}^2)\ddot{\theta}_2$$
$$= m_2 g l_{C2}\sin\theta_2 + \sin\theta_2 + m_2 l_1 l_{C2}\sin(\theta_1-\theta_2)\dot{\theta}_1^2 - c_2(\dot{\theta}_1-\dot{\theta}_2)$$

$$-(m_1 l_{C1} + m_2 l_1)\cos\theta_1 \ddot{x}_4 + (J_{C1} + m_1 l_{C1}^2 + m_2 l_1{}^2)\ddot{\theta}_1$$
$$+ m_2 l_1 l_{C2}\cos(\theta_1 - \theta_2)\ddot{\theta}_2 = m_1 g l_{C1}\sin\theta_1 + m_2 g l_1 \sin\theta_1$$
$$- m_2 l_1 l_{C2}\sin(\theta_1 - \theta_2)\dot{\theta}_2^2 - c_1\dot{\theta}_1 - c_2(\dot{\theta}_1 - \dot{\theta}_2)$$

$$(m_4 + m_1 + m_2)\ddot{x}_4 - (m_1 l_{C1} + m_2 l_1)(\mu\sin\theta_1 + \cos\theta_1)\ddot{\theta}_1$$
$$-(\mu m_2 l_{C2}\sin\theta_2 + m_2 l_{C2}\cos\theta_2)\ddot{\theta}_2$$
$$= \mu(m_1 l_{C1} + m_2 l_1)\dot{\theta}_1^2\cos\theta_1 + \mu m_2 l_{C2}\dot{\theta}_2^2\cos\theta_2$$
$$-(m_1 l_{C1} + m_2 l_1)\dot{\theta}_1^2\sin\theta_1 - m_2 l_{C2}\dot{\theta}_2^2\sin\theta_2 - k(x_4 - x_3)$$
$$-\mu(m_4 + m_1 + m_2)g - c\dot{x}_4$$

$$m_3\ddot{x}_3 = F(t) + k(x_4 - x_3) - \mu m_3 g - c\dot{x}_3 \qquad (1)$$

where $m_3 =$ mass of cart 3, $m_4 =$ mass of cart 4, $c_0 =$ friction factor, $c =$ tack coefficient, and $u(t) =$ control input.

2.2 Problem Statement

By Letting $z_1 = x_3$, $z_2 = x_4$, $z_3 = \theta_1$, $z_4 = \theta_2$, $z_5 = x_3$, $z_6 = x_4$, $z_7 = \theta_1$, and $z_8 = \theta_2$, the equation of motion in Eq. (1) can be converted in the form of a first order differential equation, $\dot{\underline{z}} = f(\underline{z}(t), u(t))$ where $\underline{z}(t) = [z_1, z_2, z_3, z_4, z_5, z_6]^T$. Then, the linear system can be expressed in the state space form as

$$\dot{\underline{z}} = A\underline{z} + Bu \text{ and } y = C\underline{z} + Du \qquad (2)$$

where u(t) = control input,
y(t) = output variable,
A = State matrix,
B = input matrix,
C = Output matrix and
D = input-to-output coupling matrix.

3. H Infinity Control Design Method

The main purpose of H-infinity design is to synthesize optimal full information controller that minimize infinity norm of the closed loop system between disturbance and input. Under the condition that information of all states and disturbance input are available for feedback [5-6]. The optimal control problem is considered as a dynamic optimization problem, mini-max problem. The objective function is defined as a cost function, J and optimization variables are disturbance, $w(t)$ and output, $y(t)$ as shown in Eq.(4). Also, the state space of the system is considered as dynamic constraints

$$J = \|G_{yw}\|_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]} \neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}} < \gamma \qquad (3)$$

$$\dot{\underline{z}} = A\underline{z} + B_u u + B_w w$$
$$y = C_y \underline{z} + D_{yu} u \qquad (4)$$

After solving the Hamiltonian equation corresponding to from Eq.(3) and Eq.(4), the suboptimal control can be determined as

$$u(t) = K(t)\underline{z}(t), \qquad (5)$$

where $K(t) = B_u^T P$ and a matrix $P(t)$ is the solution of the Riccati differential equation as shown in Eq(6).

$$PA + A^T P - P(B_u B_u^T - \gamma B_w B_w^T)P + C_y^T C_y = 0 \qquad (6)$$

for H_∞ sub optimal control under the condition that

$$A - (B_u B_u^T - \gamma^2 B_w B_w^T)P \qquad (7)$$

is stable.

The details are purposed in many textbooks Burl [5],Helton et al.[3],Tsachouridis et al. [4],and Skpgestal and Postlethwaite [6].

4. Simulation and Results

In order to show the feasibility of H infinity controller design method, the simulation of applying H infinity controller applying to stabilize the example of a linear spring connected double inverted pendulum under disturbance and compared with LQR controller design method are presented. The simulation is implemented in MATLAB software, and the simulation results of both approaches are provided and discussed as follows. The example of a linear spring connected double inverted pendulum system is defined by specifying the parameters of the system as follows:

$m_1 = 0.25$ kg, $m_2 = 0.25$ kg, $m_3 = 1.5$ kg,
$m_4 = 1.5$ kg, $c_0 = 0$, $c = 0$, $l_1 = 0.4$ m,
$J_{c1} = 0.0033$ kg·m², $l_{C1} = 0.2$ m, $c_1 = 0.05$,
$l_2 = 0.4$ m, $J_{c2} = 0.0033$ kg·m², $l_{C2} = 0.2$ m,
$c_2 = 0.05$, $k = 100$ N/m, $g = 9.81$ m/s² in appropriate SI units.

Both controller design methods are tested by disturbance signal which is bound in the form of

$$w(t) = e^{-at}\bar{A}\sin(\sqrt{b}t + \phi) \qquad (8)$$

where, $\overline{A}=1$, $a=0.8$, $b=10$, and $\phi=0$ as shown in Fig. 2. All time response of state variables of H-infinity and LQR methods are under the disturbance signal are presented in Fig. 3. The blue dash curves and magenta curves in Fig. 3 represent the time responses of all state variables corresponding to H-infinity and LQR respectively.

The H-infinity feedback control system is stable, since all poles or eigenvalues of $A-(B_uB_u^T-\gamma^2B_wB_w^T)P$ are located on the left half plane as shown in Fig. 4. The numerical solution of Riccati's Equation is shown in Eq.(9).

$$P=\begin{bmatrix} 0.0001 & -0.0001 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0001 & 0.0001 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 1.1032 & -0.5402 & -0.0000 & -0.0025 & 0.0779 & -0.0132 \\ 0.0000 & 0.0000 & -0.5402 & 0.2693 & 0.0000 & 0.0010 & -0.0376 & 0.0072 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0025 & 0.0010 & -0.0000 & 0.0006 & -0.0002 & 0.0000 \\ -0.0000 & -0.0000 & 0.0779 & -0.0376 & -0.0000 & -0.0002 & 0.0056 & -0.0008 \\ 0.0000 & 0.0000 & -0.0132 & 0.0072 & 0.0000 & 0.0000 & -0.0008 & 0.0003 \end{bmatrix} 10^5 \quad (9)$$

Eigenvalues of P, $\lambda(P)$, are

$$\lambda(P)=\{1.9252e-004, 0.0060, 0.1342, 1.0737, 15.1359, 56.7692, 408.5991, 1.3742e+005\} \quad (10)$$

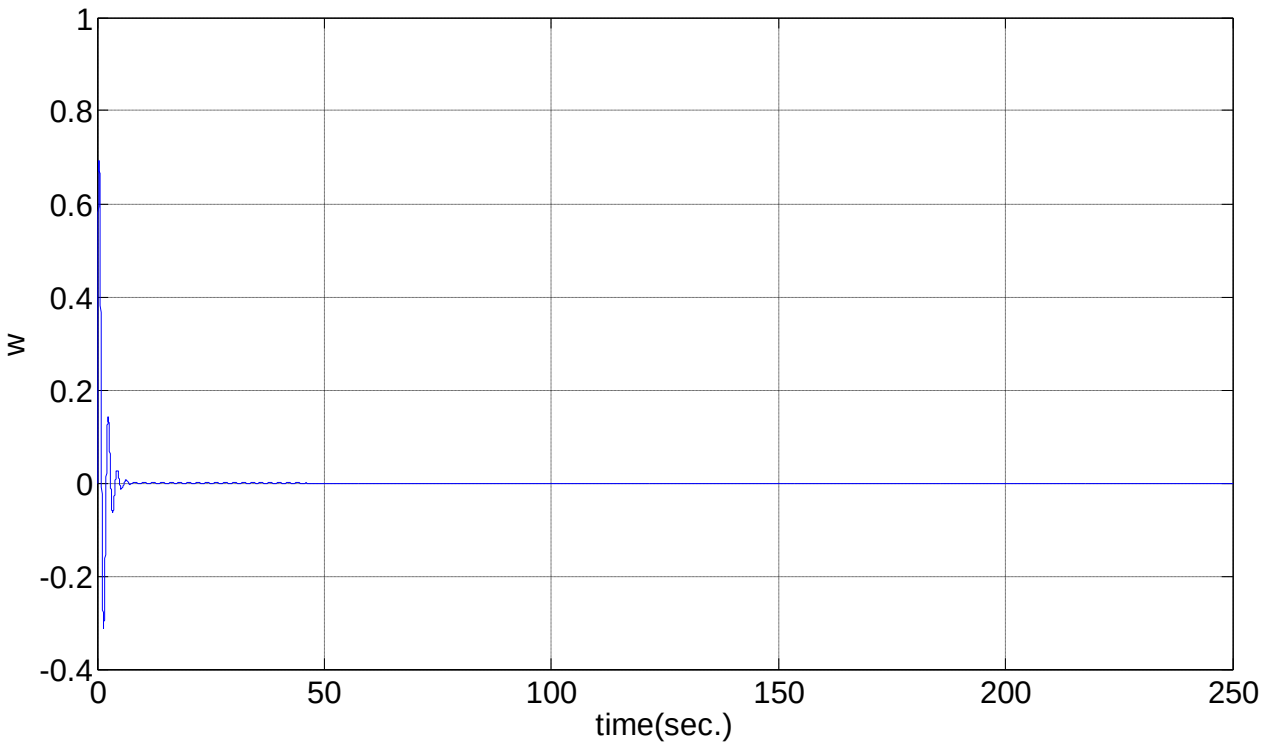

Fig. 2 A Disturbance signal

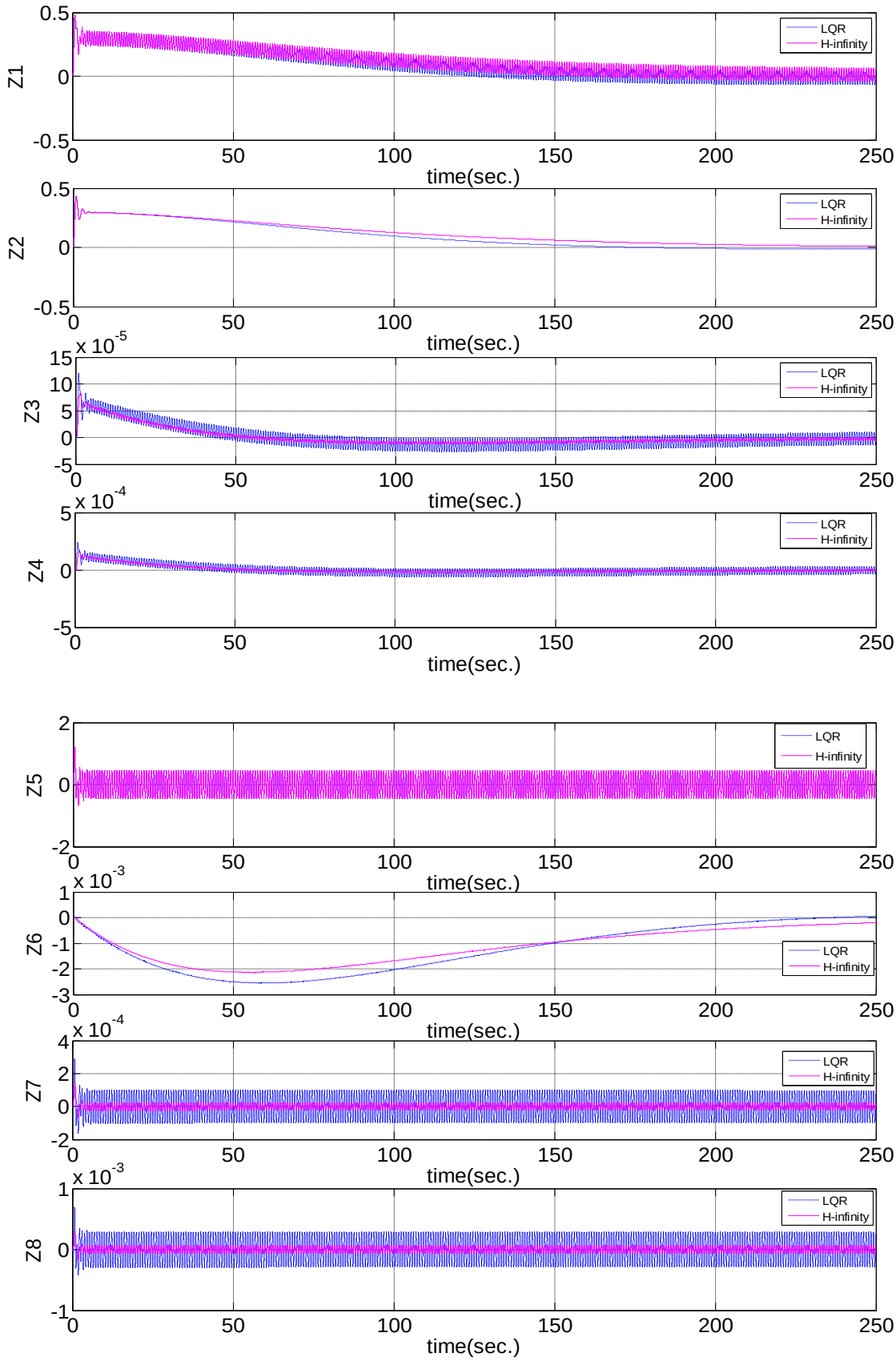

Fig. 3 Time response of state variables of Z_1 ,Z_2, ,Z_3 ,Z_4 ,Z_5 ,Z_6 ,Z_7 ,and Z_8 of the H-infinity method (in blue dash curves) and the LQR method(in magenta curves).

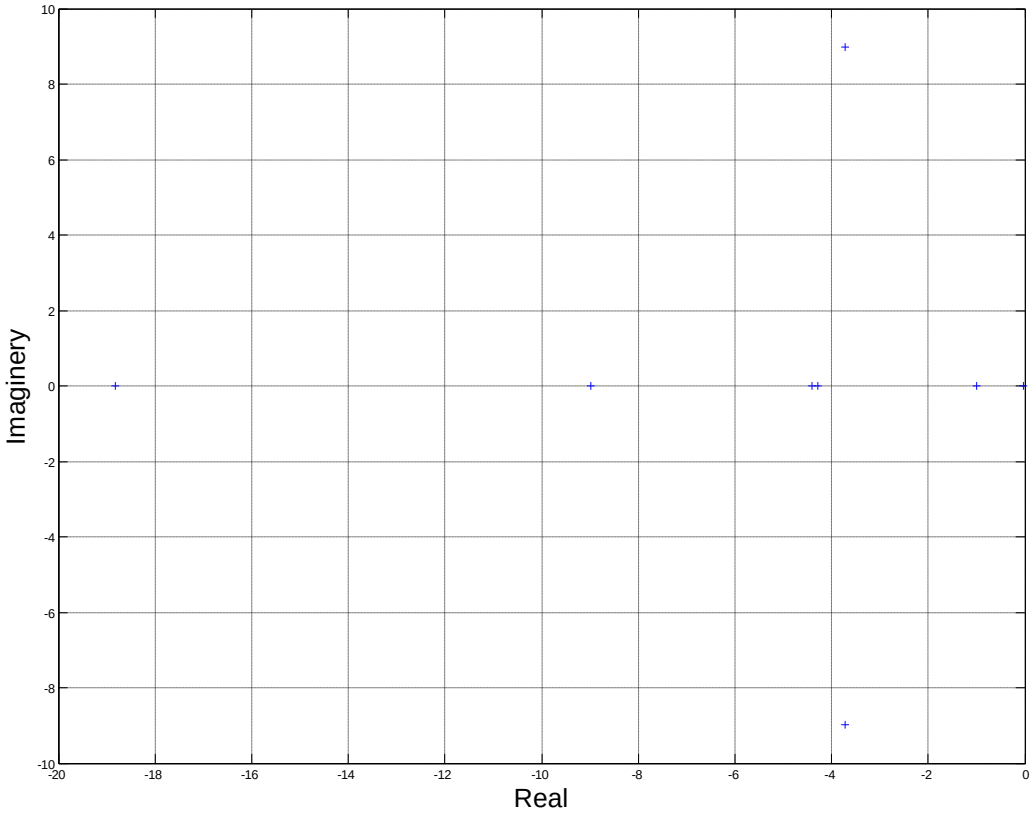

Fig. 4 Poles of the H-infinity feedback control system.

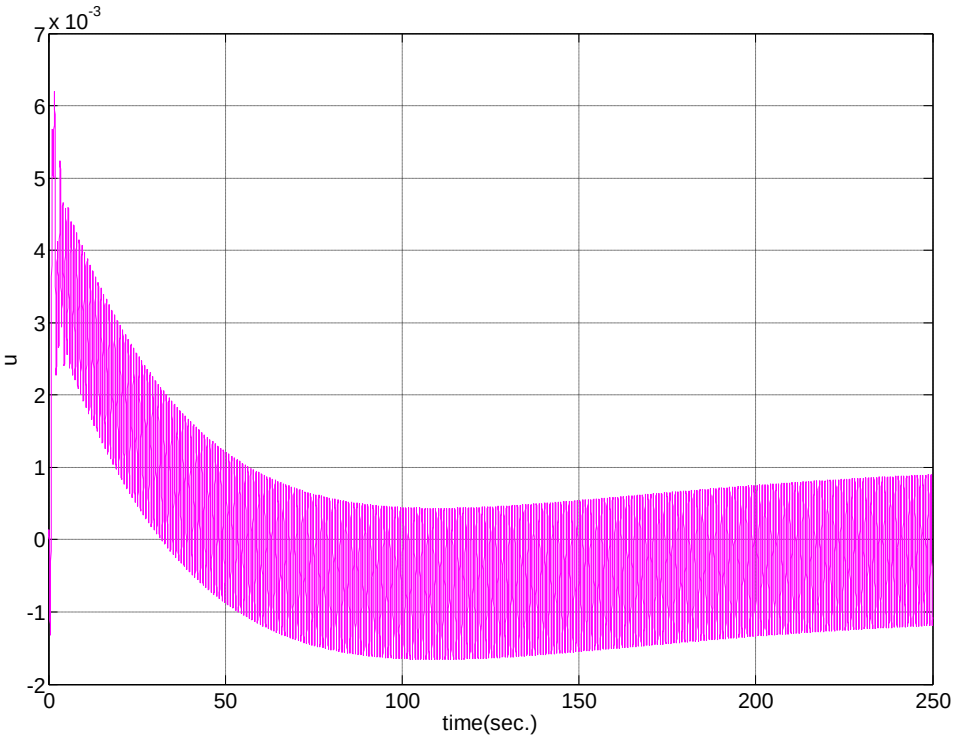

Fig. 5 A control input signal of H-infinity feedback control system.

Since the positive definite solution of Riccati differential equation exist and the system $A-(B_u B_u^T - \gamma^2 B_w B_w^T)P$ is stable, therefore, the H-infinity controller can be applied to this problem. The control input signal corresponding to H-infinity method is presented in Fig. 5.

Considering state variables of $z_3 = \theta_1$, $z_4 = \theta_2$, $z_5 = x_3$, $z_6 = x_4$, $z_7 = \theta_1$, and $z_8 = \theta_2$ at steady state, it is clear that H-infinity gives the smaller magnitude of amplitude than those of LQR. The time responses of $z_3 = \theta_1$, $z_4 = \theta_2$, $z_5 = x_3$, $z_6 = x_4$, $z_7 = \theta_1$, and $z_8 = \theta_2$, corresponding to both methods oscillate with bound about equilibrium point of zero. Even though the state variables of $z_1 = x_3$ and $z_2 = x_4$ at stead, of LQR method have smaller magnitude than those of H-infinity. The time responses of these state variables $z_1 = x_3$ of both methods oscillate with bound about equilibrium point of zero at steady and the difference between signal given by both methods are very small. The time response of state variables $z_2 = x_4$ of both methods converges to zero with very small difference in magnitude.

Using H-infinity controller, the cart with mass m_3 oscillate with about the equilibrium point with the bound of amplitude about 0.09 m, while the cart with mass m_4 converge to zero at steady state. The double inverted pendulum is at the upright position since the time response of both method have magnitude in the range of $\pm 7.6 \times 10^{-6}$ rad and $\pm 1.7 \times 10^{-5}$ rad for θ_1 and θ_2 respectively.

The main purpose is to maintain the upright positions of double inverted pendulum. Based on this aspect, the H-infinity method performs better than the LQR does, since the H-infinity method gives the smaller magnitude of time responses of $z_3 = \theta_1$ and $z_4 = \theta_2$ than that of LQR method. Therefore, it is feasible to stabilize the system by using H-infinity controller under the finite energy disturbance signal.

5. Conclusions

H-infinity controller can be used to control and stabilized a mass spring connected double inverted pendulum system under the finite energy disturbance as shown and discussed above. Comparison between H-infinity controller and LQR controller shows that H-infinity controller performs better than LQR does.

6. References

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