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Accuracy of the Scaling Law for Experimental Natural Frequencies of Rectangular Thin Plates

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Abstract

A scaling law for vibration response of rectangular isotropic plates along with a similarity requirement was derived and verified by experiment method in this study. The scaling law was derived from the governing equation of the problem, and verified with the closed form solution. Besides theoretical verification, the experiment study was conducted on model and prototype specimens. A number of nine aluminum rectangular plates with SSSF boundary conditions were tested for natural frequencies using impact test method. Accelerometer and dynamic signal analyzer were employed to measure and analyze the vibration response of the specimens. Natural frequencies of the first three vibration modes were obtained by transforming the acceleration in time domain to that of the frequency domain. The natural frequencies of the models were substituted into the scaling law to obtain the scaling natural frequencies of the prototypes, which were compared to the measured natural frequencies. From a total of 9 comparisons, the average percent discrepancy of the scaling natural frequencies is 0.25% with standard deviation of 6.4%. Natural frequencies of the prototype determined from both approaches agree with each other very well. The accuracy in this study is notably better that of the similar study on scaling law for buckling of plate. Thus, the derived scaling can be used in engineering applications, providing that the boundary conditions of the model and prototype are identical.

Keywords: Scaling Law, Natural Frequency, Similitude

1. Introduction

The similitude concept has been utilized in many engineering applications. It is very helpful for engineers to be able to replicate the behavior of the prototype using the appropriate scaled model. The concept is also very powerful for problems with complicated boundary conditions where analytical or numerical solutions are not sufficiently accurate, if not impossible. Similitude theory can be stated as [1]; "the sufficient and necessary condition of similitude between two systems is that the mathematical model of the one be related by a bi-unique transformation to that of the other." If parameters of the model and prototype have such similarity conditions, then the scaled replica can be built to duplicate the behaviors of the full-scaled system, and the results from the model experiments can be utilized to predict the behavior of the prototype.

The similitude theory have been applied to many problems in the field of structure engineering, especially vibration and buckling of plate problems which are in the interest of this study. Simitses [2] applied similitude transformation to bending, buckling, and vibration of laminated plates. The derived scaling laws and appropriate similarity requirements were successfully employed to the problems. Rezaeepazhand *et. al* [3] demonstrated a procedure for deriving the scaling law for frequency response parameter utilizing the closed form solution. Another approach of employing the similitude transformation for stability and vibration of laminated rectangular plate problem is presented in Ref.[4-6]. In those studies, the similitude transformation was applied to the governing equations of the problem directly. The advantage of this approach is that the solutions of the governing equations are not required. The obtained scaling laws were verified with the theoretical solution and found to be exact for complete similitude cases. Partial similitude cases were also investigated and recommended.

Beside theoretical work, the scaling law was also verified by experiment method. Alanpitak [7] performed buckling experiment on composite plates and shown that the buckling scaling law was accurate in most of the model-prototype pairs. The average percent discrepancy between scaling and experimental buckling loads was –5.9% with standard deviation of 8.7%. However, some pairs of model-prototype have percent discrepancy as high as 30% for complete similitude case. The uniformity of the specimens and the buckling load identification method load are cited as the probable causes of the discrepancy.

In this study, the scaling law for natural frequency of rectangular aluminum plates was derived and compared with the experiment results. It is an objective of this study to investigate the accuracy and repeatability of the scaling law and vibration measurement, and compare to the previous buckling problem study by Alanpitak [7] where the standard deviation of the percent discrepancy of the scaling buckling load is quite high. So, this study is intended to confirm that the scaling laws for structural problems are reliable and suitable to use in engineering application. Also, the experiment result from this study could indicate the accuracy of the natural frequency measurement.

2. Natural frequency of plate

The governing equation for vibration of isotropic rectangular thin plate can be written as [8];

$$\frac{\partial^4 W(x,y,t)}{\partial x^4}+2\frac{\partial^4 W(x,y,t)}{\partial x^2 \partial y^2}+\frac{\partial^4 W(x,y,t)}{\partial y^4}+\frac{\rho}{D}\frac{\partial^2 W(x,y,t)}{\partial t^2}=0 \quad (1)$$

where W is the displacement in the out-of-plane direction, ρ is the mass density of the specimen, and D is the plate bending stiffness.

Assuming that the out-of-plane displacement is separable as a function of position and time, the governing equation is reduced to

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}-\frac{\omega^2 \rho}{D}w=0 \quad (2)$$

where w is function of x and y only, i.e. $w = w(x,y)$, and ω is the frequency of the vibration.

The vibration equation, eq. (2), can be solved if the boundary conditions of the plate are known. For simple-supported plates, the analytical closed form solution is

$$\omega_{mn}=\frac{\pi}{2a^2}\sqrt{\frac{D}{\rho h}}\left(m^2+\frac{a^2}{b^2}n^2\right) \quad (3)$$

where ω_{mn} are natural frequency of the plate in Hz, a and b are plate width and length, respectively, h is specimen thickness, m and n are positive integer.

3. Scaling law for vibration of plate

The scaling law for vibration of rectangular isotropic plates is derived from the governing equation, eq.(2) by comparing the governing equation for the model and prototype systems. From both equations, the similitude invariant term, which leads to the scaling law, is obtained. Let the variables of the prototype and their corresponding model variables be related to each other as follows,

$$x_p = C_x x_m,\quad y_p = C_y y_m,\quad w_p = C_w w_m,\quad D_p = C_D D_m,$$
$$\omega_p = C_\omega \omega_m, \text{ and } \rho_p = C_\rho \rho_m$$

where subscripted p refers to prototype and subscripted m refers to model, and C_i are the scaling factors of the i parameters. To derive the similitude invariant, the governing equations of the model and prototype are written as the following,

$$\frac{\partial^4 w_m}{\partial x_m^4}+2\frac{\partial^4 w_m}{\partial x_m^2 \partial y_m^2}+\frac{\partial^4 w_m}{\partial y_m^4}-\frac{\omega_m^2 \rho_m}{D_m}w_m=0 \quad (4)$$

$$\frac{C_w}{C_x^4}\frac{\partial^4 w_m}{\partial x_m^4}+2\frac{C_w}{C_x^2 C_y^2}\frac{\partial^4 w_m}{\partial x_m^2 \partial y_m^2}+\frac{C_w}{C_y^4}\frac{\partial^4 w_m}{\partial y_m^4}-\frac{C_\omega^2 C_\rho C_w}{C_D}\frac{\omega_m^2 \rho_m}{D_m}w_m=0 \quad (5)$$

Comparing both equations, the vibration behavior of the model and prototype are similar if groups of the scaling factors in eq.(5) are all equal. This implies that eq.(5) can be reduced to eq.(4) when the scaling factor groups are canceled out. Thus, the similitude requirement is obtained as

$$\frac{1}{C_x^4}=\frac{1}{C_x^2 C_y^2}=\frac{1}{C_y^4}=\frac{C_\omega^2 C_\rho}{C_D} \quad (6)$$

By assume that the model and prototype have geometric similarity ($C_x = C_y = C_a = C_b$), the similarity requirement is simplified to,

$$\frac{C_\omega^2 C_\rho C_b^4}{C_D}=1 \quad (7)$$

Eq.(7) is the similitude invariant of the vibration of rectangular plates. This invariant can be reduced to the scaling law of plate natural frequency as,

$$\omega_p^2=\omega_m^2 C_D\frac{b_m^4 \rho_m}{b_p^4 \rho_p} \quad (8)$$

In conclusion, the derived scaling law for vibration of plate is valid for a model-prototype pair with complete geometric similarity, i.e. $C_a = C_b$ or both systems have the same aspect ratio. The scaling law can be verified with the theoretical solution shown in the previous section, as shown in Table 1. Rectangular plates with b = 250mm and aspect ratio of 1, 1.5, and 2 are selected as models, and used to predict the natural frequencies of the prototypes with b = 200 mm and 300 mm. All plates are assumed to be Al6061-T6 with E = 68.9 GPa, v = 0.35, ρ = 2.71×10^3 kg/m^3, and plate thickness h = 2 mm. The natural frequencies of the models determined from the analytical solution, eq.(3), are shown in column 2. These natural frequencies are substituted into the scaling law to predict the scaling natural frequencies of the prototypes, as presented in the “Scaling” columns. The scaling frequencies are verified with the theoretical solutions shown in column 4 and 6. It is confirmed that the natural frequencies determined from the scaling law and those from the closed form solutions are identical. Therefore, the derived scaling law for natural frequency of rectangular plate is verified, theoretically.

Next, the derived scaling law is validated by real measurements. The experiment is performed to determine the accuracy and repeatability of the approach.

Table 1. The fundamental natural frequency in Hz for Al6061-T6 specimens

Aspect ratio	Model b=250mm	Prototype			
		b = 200mm		b = 300mm	
		Theory	Scaling	Theory	Scaling
1	156.2	244.1	244.1	108.5	108.5
1.5	112.8	176.3	176.3	78.4	78.4
2	97.6	152.6	152.6	67.8	67.8

3. Experiment study

To determine the accuracy of the scaling law, rectangular aluminum thin specimens were prepared and measured for natural frequencies. The specimens were then classified as a model or a prototype. The measured natural frequencies of the models were used along with the scaling law to predict the natural frequencies of the prototypes. Then, the scaling natural frequencies were compared with the experimental natural frequencies of the prototype to find the accuracy of the scaling law.

3.1 Experiment setup and specimens

A total of nine specimens with the length and width of a and b, respectively, were tested in this study. The schematic drawing of the specimen dimension and boundary condition are shown in Fig.1. The specimen boundary conditions are simple support, as shown by a dashed line in Fig.1, on three edges and free on one of the width edge. The specimen aspect ratios (a/b) are 1, 1.5, and 2 with specimen nominal width b of 200, 250, and 300 mm., respectively. The specimens are mounted in the experiment setup and equipped with impact hammer and accelerometer as shown in Fig.2. The simply supported boundary condition is enforced by two stainless steel bars coupled on the specimen. The support is cut in the inclined direction to form a knife-edge. These supports allow the specimen to freely rotate, but restrain any out-of-plane displacement. The knife-edge supports are fixed with steel boxes by machine screws. There are additional machine screws used to push the support against the specimen surface. The assembly of steel boxes and knife-edge supports was tested for natural frequency, also, to confirm that their natural frequencies are not in the range of that of the specimens. The vibration test for natural frequency is determined by impact test. The impact hammer is used to excite the specimen. The applied impulse can be monitor by the dynamic signal analyzer. An accelerometer is placed on the specimen at a selected location to measure the plate response in term of acceleration. It is recommended that the accelerometer should not be on the node line of the vibration to avoid low response signal. If the node line is unknown or uncertain, more than one measurement is recommended. In this study, five pretests were conducted to determine a suitable accelerometer's location. Measured accelerations from the accelerometer are collected by a dynamic signal analyzer and used to determine the natural frequencies.

3.2 Data analysis

The acceleration measured in time domain is processed by a Fast Fourier Transform (FFT) algorithm using the dynamic signal analyzer. From the vibration response in frequency domain, the natural frequencies of the specimen are identified from the peak of the response. Theoretically, there are infinite number of natural frequency, however, only the first three modes are studied in this study. Fig.3 shows the vibration response measured in frequency domain obtained from the data in time domain for specimens with dimension of 30×20 and 60×30cm^2. The measured natural frequencies in Hz for the first three modes are 175, 279, 449 and 89, 110.5, 149, respectively.

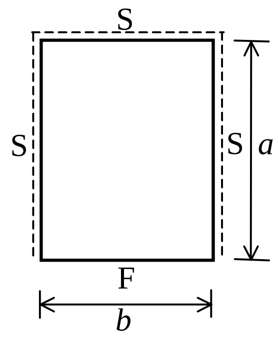


Fig. 1 Schematic drawing of rectangular specimen

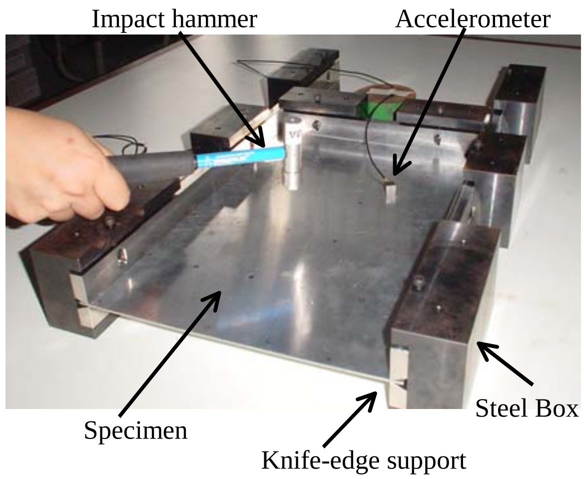
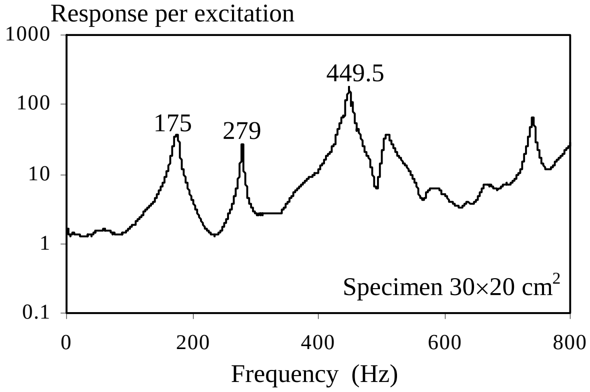


Fig. 2 Experiment setup with accelerometer and impact hammer

The experimental results similar to those of Fig. 3 can be obtained from an experiment with excitation and accelerometer locations located at particular positions. Ideally, the same natural frequencies should be acquired no matter where the excitation and accelerometer are located. In this study, a total of 25 measurements were performed on each specimen. An accelerometer was placed on a selected point, while a total of 25 excitation points was systematically varied to cover the whole specimen area. The experimental natural frequency was determined from the average of each measurement. For a specific specimen, natural frequencies from each measurement are very similar which confirms that the experiment is very repeatable.



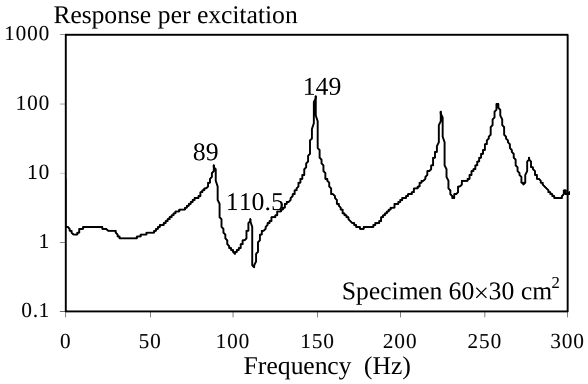


Fig. 3. Vibration response in frequency domain

4. Experiment result

From the vibration response in the frequency domain, the first three natural frequencies of each specimen are presented in Table 2. The specimens are classified into three groups of aspect ratios of 1, 1.5, and 2. It is noticed that the natural frequencies decrease when the specimen is bigger. The specimens are assumed to be a model or a prototype and used to verify the scaling law, as shown in Table 3. From three specimens with aspect ratio of 1, three comparisons of scaling and measured natural frequencies can be made. That is, as shown in column 2 and 3 of Table 3, a specimen with 25×25 dimension is set as a model, and other two specimens are prototype. The other model-prototype pair is specimen 20×20 and specimen 30×30. The other two aspect ratios can also be compared in the same approach. In Table 3, column 4 and column 5 are the measured natural frequencies of the model and prototype, respectively. The next column labeled “Scaling” is the scaling natural frequency of the prototype. These scaling natural frequencies are determined from the scaling law shown in eq.(8) using the measured natural frequencies of the model. The scaling and experimental natural frequencies shown in column 6 and 5 are compared to each other. The percent discrepancy of the scaling natural frequency is determined according to,

$$\%Dis = \frac{\omega_{Scaling} - \omega_{Exp.}}{\omega_{Exp.}} \times 100\% \qquad (9)$$

The comparisons for vibration in mode 2 and 3 are performed similar to those of mode 1, and shown in the second and third parts of the table. The average and standard deviation of percent discrepancy for each vibration mode are shown in the last two rows of Table 3. The average percent discrepancy in each mode is as high as 1.25% with the overall percent discrepancy of 0.25%. There is a case of comparison where percent discrepancy is as high as 18.7%. There is no significant difference in average percent discrepancy for each vibration mode. The overall standard deviations from 27 comparisons of percent discrepancy are 6.4%. The vibration mode 1 has the highest deviation of percent discrepancy of 9.94% while the deviation of percent discrepancy decreases in the higher modes. This implies that the measurement of natural of the high mode is accurate than that of the lower mode

5. Discussion and conclusion

Compared to the scaling law for buckling of plate studied by Alanpitak [7], the scaling law for vibration response shows a better accuracy in prediction of natural frequency. The average percent discrepancy for vibration measurement is significantly better, i.e. 0.25% compared to –5.9%. Similarly, the standard deviation of the percent discrepancy for this vibration experiment is lower than that of the buckling experiment. Experiment results of both buckling and vibration experiment do not imply that the vibration scaling law is accurate than that of the buckling problem. The difference in accuracy found in both problems is probably initiated from the difficulty in identifying the buckling point in buckling experiment which is not found in vibration experiment. The natural frequency determined from the vibration response in the frequency domain, as shown in Fig. 3, is more accurate than the buckling load determined from the buckling experiment.

In conclusion, this research derives the scaling law for vibration response of rectangular thin plates. In addition to the scaling law, the similitude requirements for two systems to behave similarly are also obtained. The scaling law is verified with the theoretical solution and found that the scaling natural frequency is exactly matched with the closed form solution. The experiment setup was built to accommodate the vibration experiment. A set of nine rectangular aluminum plates was set in the test setup with simple support on three edges and free support on one edge. The specimen was excited by impact hammer and measured for vibration response using an accelerometer. The measured acceleration response in time domain was then transformed to the data in frequency domain. The natural frequencies can be identified from the peaks of the response in frequency domain.

The experiment results were used to verify the scaling law. It is found that the average discrepancy between the scaling and experimental natural frequencies is 0.25% with 6.4% standard deviation. This discrepancy is very much lower than that of the buckling experiment studied previously. This suggests that, in nature, the natural frequency can be measured with higher accuracy and repeatability. A better agreement between the scaling and measured natural frequencies can be obtained if both systems have higher degree of similarity.

Table 2. Measured natural frequencies of the specimens SSSF boundary conditions

Aspect ratio	Size $a \times b$ (cm^2)	Mode 1 (Hz)	Mode 2 (Hz)	Mode 3 (Hz)
1	20×20	223.0	412.0	588.0
	25×25	133.5	274.5	358.0
	30×30	83.5	173.5	251.0
1.5	30×20	175.5	279.5	449.5
	37.5×25	117.5	175.5	282.0
	45×30	86.0	130.0	197.0
2	40×20	181.0	250.0	334.0
	50×25	120.5	164.0	211.5
	60×30	89.0	111.5	149.5

Table 3. Natural frequencies determined from the scaling law compared with the experimental results

Aspect ratio	Model	Prototype	ω of mode 1				ω of mode 2				ω mode 3			
			Model	Prototype			Model	Prototype			Model	Prototype		
				Exp.	Scaling	% Dis		Exp.	Scaling	% Dis		Exp.	Scaling	% Dis
1	20×20	30×30	223.0	83.5	99.1	18.7	412.0	173.5	183.1	5.5	588.0	251.0	261.3	4.1
	25×25	20×20	133.5	223.0	208.6	-6.5	274.5	412.0	428.9	4.1	358.0	588.0	559.4	-4.9
	25×25	30×30	133.5	83.5	92.7	11.0	274.5	173.5	190.6	9.9	358.0	251.0	248.6	-1.0
1.5	30×20	45×30	175.5	86.0	78.0	-9.3	279.5	130.0	124.2	-4.4	449.5	197.0	199.8	1.4
	37.5×25	30×20	117.5	175.5	183.6	4.6	175.5	279.5	274.2	-1.9	282.0	449.5	440.6	-2.0
	37.5×25	45×30	117.5	86.0	81.6	-5.1	175.5	130.0	121.9	-6.3	282.0	197.0	195.8	-0.6
2	40×20	60×30	181.0	89.0	80.4	-9.6	250.0	111.5	111.1	-0.3	334.0	149.5	148.4	-0.7
	50×25	40×20	120.5	181.0	188.3	4.0	164.0	250.0	256.3	2.5	211.5	334.0	330.5	-1.1
	50×25	60×30	120.5	89.0	83.7	-6.0	164.0	111.5	113.9	2.1	211.5	149.5	146.9	-1.8
					Avg.	0.21			Avg.	1.25			Avg.	-0.71
					Std.	9.94			Std.	5.05			Std.	2.45

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