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The 2D Boundary Element Method for Creep With Variable Loads

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Abstract
A two dimensional boundary element method (BEM) formulation based on an initial strain approach has been successfully implemented for creep problems with variable loads. The time hardening and strain hardening creep problems of a square plate and a plate with a circular hole are investigated for primary creep using isoparametric quadratic elements to model the boundary with 3-node boundary elements, and to model the interior domain with 8-node quadrilateral cells. The results of the problems are compared with the finite element solutions using MSC.Marc software and the analytical solutions where available and shown to be in good agreement.

Key Words: Boundary element method; Creep; Time hardening; Strain hardening; Isoparametric quadratic element.

1. Introduction

The boundary element method (BEM) has been widely used to analyse both elastic and time-dependent inelastic engineering problems. Telles and Brebbia [1] presented the BEM formulation based on an initial strain approach for 2-D elastoplastic problems. Linear interpolation functions were employed for the boundary elements and the internal triangular cells. The von Mises yield criterion and the Prandtl-Reuss flow rule were applied for the plastic strain increment. The problems of a perforated aluminium strip under uniaxial tension, a polystyrene crazing problem under uniaxial and biaxial tension and plate strain punch were analysed. The results were compared and agreed well with the FEM and experimental results. Lee and Fenner [2] have presented the isoparametric quadratic boundary element formulation for two-dimensional elastoplastic analysis based on an initial strain approach. The problems of uniaxial tensile behaviour, bending behaviour, internally pressurised cylinder, perforated plate in tension, and uniaxial behaviour in cyclic plasticity were analysed. The results were compared to and agreed well with the analytical solutions, experimental results and the FEM. Banerjee and Raveendra [3] have proposed the boundary element formulation based on an initial stress approach for 2-D and 3-D elastoplastic problems. The quadratic isoparametric representation was used to model the boundary elements and the volume cells. The problems of 2-D and 3-D thick cylinder and 3-D thick sphere under internal pressure, 2-D and 3-D perforated strip under tension and 2-D notched plate under axial tension were analysed. The results agreed well with the FEM and experimental results. Telles and Brebbia [4] have presented the boundary element formulation based on an initial stress approach for 2-D (plane stress and plane strain) and 3-D viscoplasticity and creep problems. Euler's formula was used for time integration. The problems of a deep beam under uniform load, a thin disc under constant external edge load and a plate under thermal shrinkage were solved and compared with the FEM and the analytical solutions showing good agreement. Cathie and Banerjee [5] have presented the 3-D boundary element method for inelastic (plasticity and creep) problems. Two approaches, initial stress and initial strain, as well as the solution algorithm were introduced. A combined creep law which included both time hardening and strain hardening creep laws has been presented. The problems of square plates with and without holes under constant uniaxial and biaxial tension and a thick cylinder under internal pressure were analysed using a power law creep function. The boundary geometry and unknowns were represented by quadratic elements. The forward difference approximation (Euler) was implemented for time integration. The results agreed well with the exact solutions.

In this paper a BE formulation for creep and time-dependent material behaviour based on an initial strain approach is presented using Norton-Bailey creep laws. The time hardening and strain hardening problems are investigated. Isoparametric quadratic elements are used for the boundary element and domain cells.

2. The 2D boundary element formulation for creep problems

2.1 Integral equation for creep problems

The BE formulation for creep is based on an initial strain approach which has the same form as that used for plasticity by replacing plastic strain rates by creep strain rates as follows (see, for example, Mukherjee [6]):

$$C_{ij}(P)\dot{u}_i(P)+\int_{\Gamma} T_{ij}(P,Q)\dot{u}_j(Q)\,d\Gamma(Q)= \int_{\Gamma} U_{ij}(P,Q)\dot{t}_j(Q)\,d\Gamma(Q)+\int_{A} W_{kij}(P,q)\dot{\varepsilon}_{ij}^{c}(q)\,dA(q) \quad (1)$$

where $\dot{u}_i$, $\dot{t}_i$ and $\dot{\varepsilon}_{ij}^{c}$ are the displacement, traction and creep strain rates, respectively. U_{ij}, T_{ij} and W_{kij} are the displacement, traction and third-order kernels, respectively, which are functions of the position of the load point P and the field point Q or the interior point q and the material properties. Γ and A are the boundary and surface of the solution domain. The algebraic expressions for the kernels U_{ij}, T_{ij} and W_{kij} can be found, for example, in Lee and Fenner [7].

2.2 Constitutive models

Two creep power laws, time hardening and strain hardening, based Prandtl-Reuss flow rule are used and can be defined as follows (see, for example, Kraus [8] and Becker and Hyde [9]): For time-hardening:

$$\dot{\varepsilon}_{ij}^{c} = \frac{3}{2} m B \left(\sigma_{eff}\right)^{(n-1)} S_{ij} t^{(m-1)} \tag{2}$$

and for strain-hardening:

$$\dot{\varepsilon}_{ij}^{c} = \frac{3}{2} m B^{\frac{1}{m}} \left(\sigma_{eff}\right)^{\frac{n-m}{m}} S_{ij} \left(\varepsilon_{eff}^{c}\right)^{\frac{m-1}{m}} \tag{3}$$

where m, B and n are material constants dependent on temperature. σ_{eff}, S_{ij} and ε_{eff}^{c} are the effective stress, the deviatoric stress and the effective creep strain, respectively. The equation (3) can be written in the same form as equation (2) using the effective time, t_{eff}, as follows:

$$\dot{\varepsilon}_{ij}^{c} = \frac{3}{2} m B \left(\sigma_{eff}\right)^{(n-1)} S_{ij} t_{eff}^{(m-1)} \tag{4}$$

where

$$t_{eff} = \left[\frac{\varepsilon_{eff}^{c}}{B \sigma_{eff}^{n}}\right]^{\frac{1}{m}} \tag{5}$$

The equations (2) and equation (4) can be used for primary creep where m<1 and for secondary creep where m = 1. The response of creep strain versus time for time and strain hardening is shown in solid line in Figure 1.

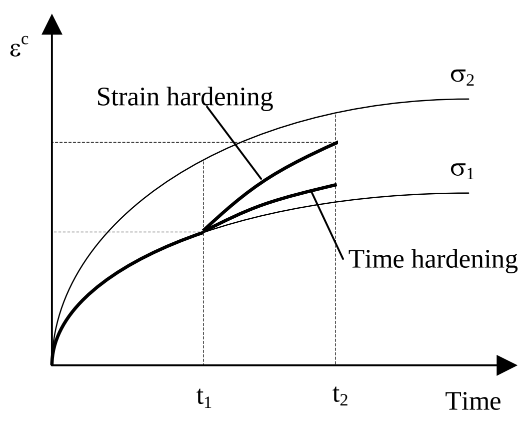

Figure 1. Time and strain hardening assumptions.

3. Numerical implementation.

To perform the integration in equation (1) numerically, the boundary and domain must be divided into a number of boundary elements and domain cells. It is convenient to use a new coordinate system that is local to the element using an intrinsic variable ξ with its origin at the midpoint node and values –1 and +1 at the end nodes. Figure 2 shows a typical three-node boundary element and a typical eight-node quadrilateral domain cell.

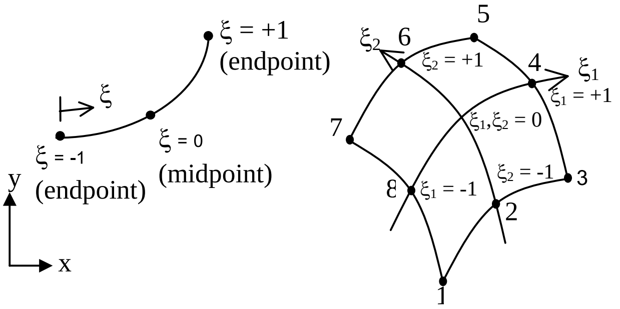

Figure 2. Isoparametric quadratic boundary element and domain cell.

Since isoparametric quadratic elements are used for the boundary and domain elements, the geometry and unknown variables have the same order and can be described using the appropriate shape functions. Therefore, the geometry and unknown variables on the boundary can be written as follows:

$$\begin{aligned} x_i(\xi) &= \sum_{c=1}^{3} N_c(\xi)(x_i)_c \\ \dot{u}_i(\xi) &= \sum_{c=1}^{3} N_c(\xi)(\dot{u}_i)_c \\ \dot{t}_i(\xi) &= \sum_{c=1}^{3} N_c(\xi)(\dot{t}_i)_c \end{aligned} \tag{6}$$

where $N_c(\xi)$ is the boundary quadratic shape function and can be found in Chandenduang [10, 11].

For the domain cells, two-dimensional quadratic shape functions are used as follows:

$$\begin{aligned} x_i(\xi_1,\xi_2) &= \sum_{c=1}^{8} N_c(\xi_1,\xi_2)(x_i)_c \\ \dot{u}_i(\xi_1,\xi_2) &= \sum_{c=1}^{8} N_c(\xi_1,\xi_2)(\dot{u}_i)_c \\ \dot{t}_i(\xi_1,\xi_2) &= \sum_{c=1}^{8} N_c(\xi_1,\xi_2)(\dot{t}_i)_c \end{aligned} \tag{7}$$

where the domain quadratic shape functions, $N_c(\xi_1,\xi_2)$, can be found in Chandenduang [10, 11]:

The integral equation (1) can be discretised into boundary elements and domain cells, and written in terms of the local coordinates as follows:

$$C_{ij}(P)\dot{u}_i(P)+\sum_{m=1}^{M}\sum_{c=1}^{3}\dot{u}_j(Q)\int_{-1}^{+1}T_{ij}(P,Q)N_c(\xi)J(\xi)d\xi =$$
$$\sum_{m=1}^{M}\sum_{c=1}^{3}\dot{t}_j(Q)\int_{-1}^{+1}U_{ij}(P,Q)N_c(\xi)J(\xi)d\xi + \qquad (8)$$
$$\sum_{d=1}^{D}\sum_{c=1}^{8}\dot{\varepsilon}_{ij}^{c}(q)\int_{-1}^{+1}\int_{-1}^{+1}W_{kij}(P,q)N_c(\xi_1,\xi_2)J(\xi_1,\xi_2)d\xi_1 d\xi_2$$

where M is the total number of the boundary elements and D is the total number of the domain cells. c is a node counter form 1 to 3 for boundary elements and 1 to 8 for domain cells. $J(\xi)$ and $J(\xi_1,\xi_2)$ are the Jacobians of transformation.

Taking each boundary node in turn as the load point P and performing the integrations, a set of linear algebraic equations can be written as follows:

$$[A][\dot{u}]=[B][\dot{t}]+[W][\dot{\varepsilon}^c] \qquad (9)$$

where the matrices [A], [B] and [W] contain the integrals of the kernels T_{ij}, U_{ij}, and W_{kij}, respectively. For two-dimensional problems, if the total number of boundary nodes is N and the total number of the domain cell points is H, then the solution matrices [A] and [B] will be square matrices of size 2N x 2N, whereas the matrix [W] will be a rectangular matrix of size 2N x 3H. Unlike the FE method, all BE matrices are fully populated.

The parameter $C_{ij}(P)$ contributes only to the diagonal coefficients of the [A] matrix (i.e. when P is equal to Q). When the points P and Q do not coincide, the standard Gaussian quadrature can be used.

4. Convergence criterion

The Euler method is used to update the variables at each time step as follows:

$$y_{i+1} = y_i + \Delta t_i \dot{y}_i \qquad (10)$$

where y represents the variable to be updated and Δt is the time step.

Although it is relatively simple to implement, the Euler method is a very slow process if a constant time step is employed. To improve the convergence rate, an automatic time step control which will automatically select the next time step for the next calculation is implemented. The main idea is to compare the error, e, at each time step, with the two predefined errors or tolerances, the maximum error, e_{max}, and the minimum error, e_{min}, as follows:

(i) If $e > e_{max}$, the current time step is reduced by a factor of less than 1.0 and the analysis is repeated.

(ii) If $e_{max} > e > e_{min}$, the current time step is used for the next calculation.

(iii) If $e_{min} > e$, the current time step is increased by a factor of greater than 1.0 for the next calculation.

The creep strain error which occurs in each time step can be defined as follows (see, e.g. Mukherjee [6]):

$$e = \frac{\left|\Delta t_i(\dot{\varepsilon}_i^c - \dot{\varepsilon}_{i-1}^c)\right|}{\left|\varepsilon_i^c\right|} \qquad (11)$$

where $\dot{\varepsilon}_i^c$ is the creep strain rate at i th step and ε_i^c is the total creep strain. Note that the stress rate can alternatively be used in equation (11) instead of the creep strain rate. In this paper the norm of the error of the effective creep strain is used and defined as follows:

$$e = \sqrt{\sum_{i=1}^{n}(e_i)^2} \qquad (12)$$

The BE algorithm for creep can be found in Chandenduang [10, 11].

5. Creep examples

All tests are performed for 100 hours using the automatic time step control with the maximum and minimum stress tolerances of 10^{-3} and 10-4, respectively. The initial time step of 10^{-3} hour and 6 integration points are used. The results are compared with analytical (Becker and Hyde [9]) and finite element (MSC.Marc [12]) solutions.

5.1 Square plate

Two cases involving a square plate under tension are tested. These tests are primary creep and plane stress conditions. The dimensions of the square plate are 100 mm x 100 mm. The boundary and domain are divided into 8 boundary elements and 4 cell, respectively, as shown in Figure 3. The material properties and creep parameters (based on stress in MPa, time in hour) are as follows:

Young’s Modulus (E) = $200\text{x}10^3$ MPa
Poisson’s Ratio (ν) = 0.3
B = $3.125\text{x}10^{-16}$
m = 0.5 for primary creep
n = 5

The boundary conditions are as follows:

u_y = 0 along line ab
u_x = 0 along line ad

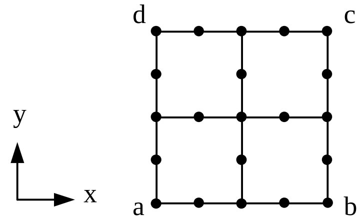

Figure 3. BE and FE mesh for the square plate (8 boundary elements and 4 cells).

Details of the tests are listed below:

1. TEST1. The square plate is subjected to a uniaxial variable constant tensile stress of 200 N/mm^2 and 250 N/mm^2 in the x-direction. The test is performed for 100 hours for the first applied stress of 200 N/mm^2 and for another 100 hours for the second applied stress of 250 N/mm^2. The automatic time step control with the maximum and minimum creep strain tolerances of 10^{-3} and 10^{-4}, respectively, is used. The initial time step of 10^{-3} hour and 6 integration points are employed. Both time hardening and strain hardening are applied. The creep strains in the x-direction are plotted against time and shown in Figure 4. The results are in good agreement with analytical solutions for both creep laws with the error being less than 0.5% and agree well with the finite element solutions.

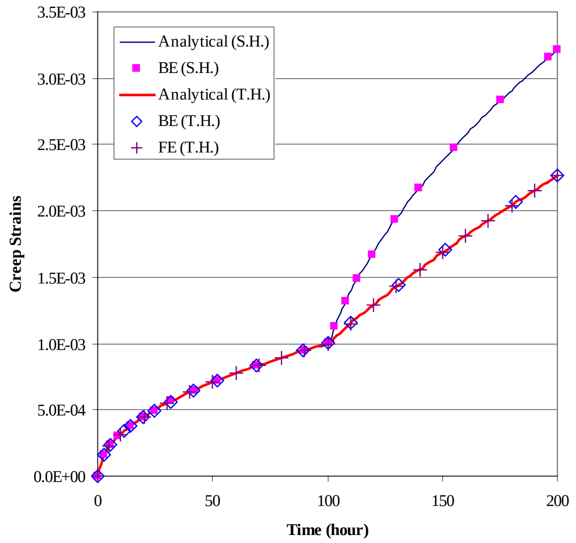

Figure 4. The BE solutions of the square plate under a uniaxial tensile stress.

2. TEST2. The square plate is subjected to a biaxial variable constant tensile stress of 200 N/mm^2 and 250 N/mm^2 in the x- and y-direction. The test is performed for 100 hours for the first applied stresses of 200 N/mm^2 and for another 100 hours for the second applied stresses of 250 N/mm^2. The automatic time step control with the maximum and minimum creep strain tolerances of 10^{-3} and 10^{-4}, respectively, is used. The initial time step of 10^{-3} hour and 6 integration points are employed. Both time hardening and strain hardening are applied. The creep strains in the x-direction are plotted against time and shown in Figure 5. The results are in good agreement with analytical solutions for both creep laws with the error being less than 0.3% and agree well with the finite element solutions.

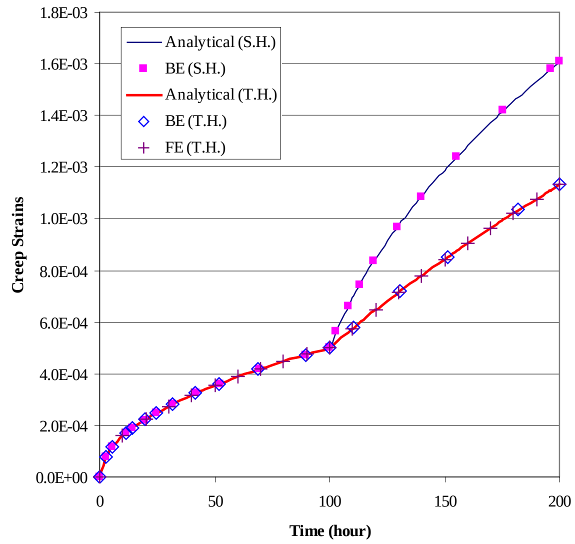

Figure 5. The BE solutions of the square plate under biaxial tensile stresses.

5.2 Square plate with a circular hole

A square plate with a circular hole at the center is analysed. Because of symmetry, only a quarter of the plate is used. The quarter of the plate with a circular hole has the dimensions of 10 mm x 10 mm with a hole of a radius of 3 mm. The boundary and domain are discretised into 28 boundary elements and 48 cells, respectively, as shown in Figure 6. The boundary conditions are as follows:

$u_y = 0$ along line ab.
$u_x = 0$ along line de.

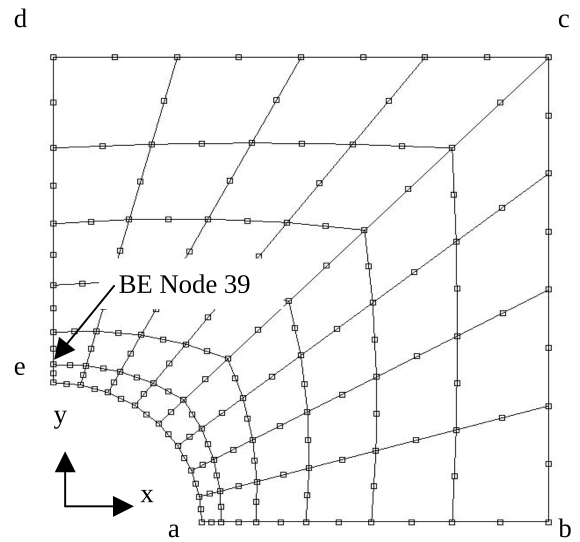

Figure 6. BE and FE mesh for the square plate with a circular hole (28 boundary elements and 48 cells).

The material properties and the creep parameters are the same as those used in the square plate tests. The plate is subjected to a uniaxial variable constant tensile stress of 40 N/mm^2 and 50 N/mm^2 in the x-direction. The test is performed for 100 hours for the first applied stress of 40

N/mm^2 and for another 100 hours for the second applied stress of 50 N/mm^2. The creep strains in the x-direction at BE node 39 are plotted against time and shown in Figure 7. The results agree well with the finite element solutions.

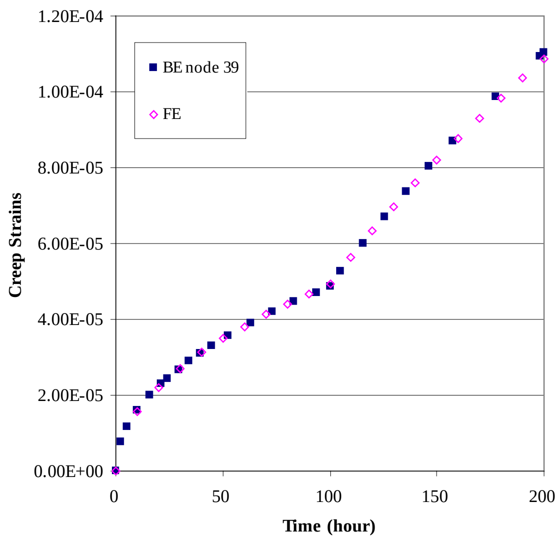

Figure 7. The BE solutions of the square plate with a circular hole under a uniaxial tensile stress.

6. Conclusion

The 2-D boundary element method for creep problems using isoparametric quadratic elements is successfully applied to solve the problems of the square plates and the plate with a circular hole. Two creep power laws, time-hardening and strain-hardening, are used to characterise materials. The results are compared with the analytical solutions and the finite element solutions using MSC.Marc and show good agreement.

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