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# An LMI Approach to Design Robust $\mathcal{H}_\infty$ Controller with Pole-Clustering Constraints

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**Abstract**

The aim of this paper is to investigate the problem of robust $\mathcal{H}_\infty$ controller design with pole-clustering or circular pole constraints. The problem we address is to design an output feedback controller such that, for all admissible parameter uncertainties, the closed-loop system satisfies not only the prespecified $\mathcal{H}_\infty$ norm constraint on the transfer function from the disturbance input to the system output, but also the prespecified circular pole constraint on the closed-loop system matrix. The necessary and sufficient conditions for the existence of desired controllers are derived in terms of a linear maitrix inequality (LMI). From the simulation results, the system responses with the proposed controller and the standard $\mathcal{H}_\infty$ controller are compared.

**Keywords**: robust $\mathcal{H}_\infty$ control, robust pole placement.

## 1 Introduction

In control sysetms design, it is currently desirable to design controller achieving robust stability due to parametric uncertainties, especially in the study of robust $\mathcal{H}_\infty$ control problem whose the objective is to design controllers such that the closed-loop system is stable and the $\mathcal{H}_\infty$ norm of a specified closed-loop transfer function is minimized. Athough the $\mathcal{H}_\infty$ control problem can be regarded as robustness against exogenous signal uncertainty, in the case when parameter uncertainty appears in the plant modelling, robust behavior on $\mathcal{H}_\infty$ performance as well as stability cannot be guaranteed by standard $\mathcal{H}_\infty$ control. This lead to the study of robust $\mathcal{H}_\infty$ control problem. However, both the standard $\mathcal{H}_\infty$ control and robust $\mathcal{H}_\infty$ control are little concerned with the transient behavior of the closed-loop system such as [9, 10] and [12].

In order to improve the transient behavior, it is well-known that the pole location is directly related to the dynamical characteristics of linear system such as damping ratios, natural and damped natural frequencies. Therefore, it is also desired to construct control systems to achieve better transient performance as well as robust stability simultaneously. A more practical way is to place the closed-loop poles in a suitable region of the complex plane, especially in circular region. Hence, it may be concluded that the closed-loop poles in a specified region guarantees both stabiltiy and the transient performance such as settling time, maximum overshoot, and rise time. For the closed-loop pole placement in a specified region, there has been continuing interest in designing controller in both nominal and uncertain systems. Many researchers have investigated in this problem such as in [4] for systems without uncertainties and in [3] for systems with uncertainties.

However, there have recently been a few reports in [6, 7] that combine robust $\mathcal{H}_\infty$ control with pole-clustering constraints not only to assure in robust stability, but also to improve better transient performance for power systems.

The rest of this paper is organized as follows. In next section, a statement of problem is provided. In Section 3, the necessary and sufficient conditions for the existence of robust $\mathcal{H}_\infty$ controller design with circular pole constraints are derived in terms of a linear matrix inequality (LMI) in order to design a desired controller. In Section 4, an illustrative example is considered to the feasibility of this proposed method and also compared with the standard robust $\mathcal{H}_\infty$ controller. From the simulation results, the system reponses with the proposed controller are better transient responses. Finally, we conclude in Section 5.

## 2 Problem Statement

We are interested in an uncertain system which has the following form:

$$\begin{aligned} \dot{x}(t) &= (A+\Delta A)x(t) + (B_u + \Delta B_u)u(t) + (F_w + \Delta F_w)w(t) \\ &= \tilde{A}x(t) + \tilde{B}_u u(t) + \tilde{F}_w w(t) \qquad (1) \\ y(t) &= Cx(t) \qquad (2) \end{aligned}$$

where $x(t) \in \Re^n$ is the state-vector of the system, $u(t) \in \Re^{n_u}$ is the control-vector of the system, $y(t) \in \Re^{n_y}$ is the output-vector, and $w(t) \in \Re^{n_w}$ is the disturbance vector of the system. The system matrices $\tilde{A}$, $\tilde{B}_u$, $\tilde{F}_w$, and $C$ have their proper dimensions and are assumed to be completely controllable and observable. The matrices $\Delta A$, $\Delta B_u$, and $\Delta F_w$ represent the parametric perturbation in the system state matrix, control input matrix, and disturbance input matrix, respectively, which are assumed to be the following norm-bounded uncertainty form:

$$\begin{pmatrix} \Delta A & \Delta B_u & \Delta F_w \end{pmatrix} = H_1 \Delta \begin{pmatrix} E & E_u & E_w \end{pmatrix} \qquad (3)$$

where $\Delta \in \Re^{k\times l}$ is an uncertain matrix bound by

$$\Delta' \Delta \leq I \qquad (4)$$

and $H_1$, $E$, $E_u$, and $E_w$ are the constant matrices of appropriate dimensions which specify how the elements of the nominal matrices $A$, $B_w$, and $F_w$ are affected by the $\Delta$. Also, $\Delta A$, $\Delta B_u$, and $\Delta F_w$ are said to be admissible, if both (3) and (4) hold.

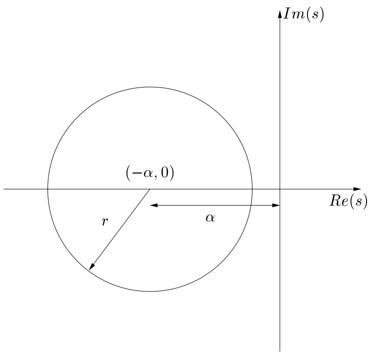


Figure 1: $\mathcal{D}(\alpha, r)$ Region

The problem of interest can be formulated in determining a linear output feedback controller $\mathcal{K}(s)$ in (21) such that the following performance criteria or design requirement are simultaneously achieved.

1. All closed-loop poles are assigned in a stable disk region $\mathcal{D}(\alpha, r)$ in the complex plane with the center at $-\alpha + j0$ ($\alpha > 0$) and the radius $r$ ($r < \alpha$) in Figure 1.
2. The $\mathcal{H}_\infty$ norm of the closed-loop transfer function $\mathcal{T}_{yw}(s)$ from $w(t)$ to $y(t)$ meets the constraint $\|\mathcal{T}_{yw}(s)\|_\infty \leq \gamma$ where $\gamma$ is a given constant and $\mathcal{T}_{yw}(s) = C_{cl}(\Delta)[sI - A_{cl}(\Delta)]^{-1} B_{cl}(\Delta)$ in (22) and (23).

## 3 Main Results

### 3.1 System Analysis

In this section, we first provide some important lemmas which will be useful in the derivation of our main results.

**Lemma 1 (Garcia and Bernussou [3])** *Let $A \in \Re^{n\times n}$ be a given matrix. Then all the poles of the closed-loop system are located with a given circular region $\mathcal{D}(\alpha, r)$, i.e., $\lambda(A) \subset \mathcal{D}(\alpha, r)$, if and only if there exists $Q > 0$ such that*

$$A_r' Q A_r - Q < 0$$

*where $A_r = (A + \alpha I)/r$.*

**Lemma 2** *Given a constant $\gamma > 0$ and a disk $\mathcal{D}(\alpha, r)$, both requirements 1. and 2. are satisfied if the following matrix inequality has a positive matrix $Q > 0$ such that*

$$\begin{pmatrix} -r^2 Q + \alpha(C'C + \gamma^{-2} Q F_w F_w' Q) & \star \\ A_\alpha & -Q^{-1} \end{pmatrix} < 0 \qquad (5)$$

*where $A_\alpha = A + \alpha I$. In addition, from a Schur complement, (5) can be rewritten as:*

$$\begin{pmatrix} -r^2 Q & \star & \star & \star \\ A_\alpha & -Q^{-1} & \star & \star \\ \sqrt{\alpha} C & 0 & -I & \star \\ \sqrt{\alpha} F_w' Q & 0 & 0 & -\gamma^2 I \end{pmatrix} < 0. \qquad (6)$$

**Proof:** By using a Schur complement [1], we can get the quadratic matrix inequality from (5) as follows.

$$A'QA + (\alpha^2 - r^2)Q + \alpha(A'Q + QA + \gamma^{-2} Q F_w F_w' Q + C'C) < 0 \qquad (7)$$

or

$$A_\alpha' Q A_\alpha - r^2 Q + \alpha(\gamma^{-2} Q F_w F_w' Q + C'C) < 0 \qquad (8)$$

It is easy to show that the circular pole requirement (1) will be met by using Lemma 1 as follows:

$$Q - A_r' Q A_r > \frac{\alpha}{r^2}(\gamma^{-2} Q F_w F_w' Q + C'C) > 0 \qquad (9)$$

where $F_w$ is of full row rank. Next, we can rearrange (7) as follows:

$$A'Q + QA + \gamma^{-2} Q F_w F_w' Q + C'C + \Sigma < 0 \qquad (10)$$

where $\Sigma = \alpha^{-1}[A'QA + (\alpha^2 - r^2)Q] > 0$. To show that the requirement 2. is also met, this proof of $\|\mathcal{T}_{yw}(s)\|_\infty \leq \gamma$ is completely similar to that of Theorem 1 of Wang [9]. ∎

**Remark 1** *From Lemma 2, we can alternatively construct another LMI condition, as shown in (11), which is equivalent to the LMI condition in (5) and (6).*

$$\begin{pmatrix} \alpha(A'Q + QA) & \star & \star & \star & \star \\ A & -Q^{-1} & \star & \star & \star \\ \beta Q & 0 & -Q & \star & \star \\ \sqrt{\alpha} C & 0 & 0 & -I & \star \\ \sqrt{\alpha} F_w' Q & 0 & 0 & 0 & -\gamma^2 I \end{pmatrix} < 0 \qquad (11)$$

*where $\beta = \sqrt{\alpha^2 - r^2}$.*

**Lemma 3 (Xie [11])** *Given matrices $G, H$ and $E$ of appropriate dimensions and $G = G'$, then*

$$G + H\Delta E + E'\Delta' H' < 0$$

*holds for any admissible uncertain matrix $\Delta$ satisfying $\Delta'\Delta \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that*

$$G + \epsilon H H' + \epsilon^{-1} E' E < 0.$$

**Definition 1** *The unforced uncertain system in (1) (setting $u(t) = 0$) is satisfied with the above performance criteria, if there exists $Q > 0$ such that*

$$\begin{pmatrix} -r^2 Q + \alpha(C'C + \gamma^{-2} Q \tilde{F}_w \tilde{F}_w' Q) & \star \\ \tilde{A}_\alpha & -Q^{-1} \end{pmatrix} < 0. \qquad (12)$$

If these uncertainties can be represented in (3) according to Definition 1, the necessary and sufficient LMI conditions are stated as follows:

**Theorem 1** *As for the unforced uncertain linear system, the desired circular pole region $\mathcal{D}(\alpha, r)$ and the $\mathcal{H}_\infty$ norm bound constraint $\gamma > 0$ on attenuation of disturbance are given. The system (1) is satisfied with requirement 1. and 2. if and only if there exists a scalar $\epsilon > 0$ and a symmetrical positive definite matrix $P \in \Re^{n\times n}$ such that*

$$\begin{pmatrix} -r^2 P & P A_\alpha' & \star & \star & \star & \star & \star & \star \\ A_\alpha P & -P & \star & \star & \star & \star & \star & \star \\ \sqrt{\alpha} C P & 0 & -I & \star & \star & \star & \star & \star \\ \sqrt{\alpha} F_w' & 0 & 0 & -\gamma^2 I & \star & \star & \star & \star \\ 0 & 0 & 0 & \sqrt{\alpha} E_w & -\epsilon I & \star & \star & \star \\ EP & 0 & 0 & 0 & 0 & -\epsilon I & \star & \star \\ \epsilon H_1' & 0 & 0 & 0 & 0 & 0 & -\epsilon I & \star \\ 0 & \epsilon H_1' & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{pmatrix} < 0. \qquad (13)$$

**Proof:** We begin with pre- and post-multiplying (12) by the matrix $\mathcal{U}'$ and $\mathcal{U}$, respectively, where $\mathcal{U} = \mathrm{diag}\{P, I\}$. Hence, we have:

$$\begin{pmatrix} \tilde{\mathcal{P}} & \star \\ \tilde{A}_\alpha P & -P \end{pmatrix} = \mathcal{U} \begin{pmatrix} \tilde{\mathcal{Q}} & \star \\ \tilde{A}_\alpha & -Q^{-1} \end{pmatrix} \mathcal{U} < 0 \qquad (14)$$

where $\tilde{\mathcal{P}} = -r^2 P + \alpha(PC'CP + \gamma^{-2}\tilde{F}_w \tilde{F}_w')$, $\tilde{\mathcal{Q}} = -r^2 Q + \alpha(C'C + \gamma^{-2} Q\tilde{F}_w \tilde{F}_w' Q)$, and $P = Q^{-1}$.

From Definition 1, we need to show that (13) is equivalent to (14). Based on (4), (14) can be rewritten as:

$$\begin{pmatrix} -r^2 P & \star & \star & \star \\ A_\alpha P & -P & \star & \star \\ \sqrt{\alpha} CP & 0 & -I & \star \\ \sqrt{\alpha} F_w' & 0 & 0 & -\gamma^2 I \end{pmatrix} + \mathcal{H}\tilde{\Delta}\mathcal{E} + (\mathcal{H}\tilde{\Delta}\mathcal{E})' < 0 \qquad (15)$$

where

$$\begin{aligned} \mathcal{H} &= \begin{pmatrix} H_1 & 0 \\ 0 & H_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\Delta} = \begin{pmatrix} \Delta & \star \\ 0 & \Delta \end{pmatrix}, \\ \mathcal{E} &= \begin{pmatrix} 0 & 0 & 0 & \sqrt{\alpha} E_w \\ EP & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

By using Lemma 3, the matrix inequality (15) holds for all $\Delta$ satisfying with (4) if and only if there exists a matrix $P > 0$ and a scalar $\epsilon > 0$ such that

$$\begin{pmatrix} -r^2 P & PA_\alpha' & \sqrt{\alpha} PC' & \sqrt{\alpha} F_w \\ A_\alpha P & -P & 0 & 0 \\ \sqrt{\alpha} CP & 0 & -I & 0 \\ \sqrt{\alpha} F_w' & 0 & 0 & -\gamma^2 I \end{pmatrix} + \epsilon \mathcal{H}\mathcal{H}' + \epsilon^{-1}\mathcal{E}'\mathcal{E} < 0 \qquad (16)$$

or

$$\begin{pmatrix} -r^2 P & \star & \star & \star \\ A_\alpha P & -P & \star & \star \\ \sqrt{\alpha} CP & 0 & -I & \star \\ \sqrt{\alpha} F_w' & 0 & 0 & -\gamma^2 I \end{pmatrix} - \mathcal{K}'\mathcal{L}^{-1}\mathcal{K} < 0 \qquad (17)$$

where

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{\alpha} E_w \\ EP & 0 & 0 & 0 \\ \epsilon H_1' & 0 & 0 & 0 \\ 0 & \epsilon H_1' & 0 & 0 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} -\epsilon I & \star & \star & \star \\ 0 & -\epsilon I & \star & \star \\ 0 & 0 & -\epsilon I & \star \\ 0 & 0 & 0 & -\epsilon I \end{pmatrix}.$$

Then using the Schur complement, it is straightforward to verify that this inequality (17) is equivalent to (13). ∎

### 3.2 Controller Design

For the design problem, we consider the following norm-bounded uncertain systems (1). The uncertainties are described by

$$\begin{pmatrix} \Delta A & \Delta B_u & \Delta F_w \\ \Delta C & * & * \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \Delta \begin{pmatrix} E & E_u & E_w \end{pmatrix} \qquad (18)$$

where $\Delta C$ is assumed to be used in practice and $*$ are neglected. Next, our approach for robust $\mathcal{H}_\infty$ controller with pole-clustering constraints is proposed by using Theorem 1.

We consider the uncertain system in (1) and desire to place all closed-loop poles of uncertain systems in $\mathcal{D}(\alpha, r)$ region and satisfy the prespecified $\mathcal{H}_\infty$ norm constraint simultaneously via a full order output feedback controller. Hence, the state space equations of the desired controller can be shown as follows:

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) & (19) \\ u(t) &= C_K x_K(t) & (20) \end{aligned}$$

where $x_K(t) \in \Re^{n_K \times n_K}$ is the state of the controller, and $A_K, B_K$, and $C_K$ are matrices with the appropriate dimensions that can be determined. For a dynamic output feedback controller, we denote its transfer function as follows.

$$\mathcal{K}(s) = C_K(sI - A_K)^{-1} B_K \qquad (21)$$

Hence, the overall closed-loop system is given by:

$$\begin{aligned} \dot{x}_{cl}(t) &= A_{cl}(\Delta) x_{cl}(t) + B_{cl}(\Delta) w(t) & (22) \\ y(t) &= C_{cl}(\Delta) x_{cl}(t) & (23) \end{aligned}$$

where

$$\begin{pmatrix} A_{cl}(\Delta) & B_{cl}(\Delta) \\ C_{cl}(\Delta) & * \end{pmatrix} = \begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & * \end{pmatrix} + \bar{H}_1 \Delta \begin{pmatrix} \tilde{E} & E_w \end{pmatrix},$$

$$\begin{aligned} x_{cl}(t) &= \begin{pmatrix} x(t) \\ x_K(t) \end{pmatrix}, A_{cl} = \begin{pmatrix} A & B_u C_K \\ B_K C & A_K \end{pmatrix}, \\ \bar{H}_1 &= \begin{pmatrix} H_1 \\ B_K H_2 \end{pmatrix}, \tilde{E} = \begin{pmatrix} E & E_u C_K \end{pmatrix}, \\ B_{cl} &= \begin{pmatrix} F_w \\ 0 \end{pmatrix}, C_{cl} = \begin{pmatrix} C & 0 \end{pmatrix}. \end{aligned}$$

**Theorem 2** *As for the uncertain linear systems (1), the desired circular pole region $\mathcal{D}(\alpha, r)$ and the $\mathcal{H}_\infty$ norm bound constraint $\gamma > 0$ on the disturbance rejection are given. The closed-loop system can acheive the expected performance requirement 1. and 2. if and only if there exist $X$, $Y$, $\bar{A}, \bar{B}, \bar{C}$, and a scalar $\epsilon > 0$ such that*

$$\begin{pmatrix} \Omega_{11} & \star & \star & \star & \star & \star & \star & \star \\ \Omega_{21} & \Omega_{22} & \star & \star & \star & \star & \star & \star \\ \Omega_{31} & 0 & -I & \star & \star & \star & \star & \star \\ \Omega_{41} & 0 & 0 & -\gamma^2 I & \star & \star & \star & \star \\ 0 & 0 & 0 & \sqrt{\alpha} E_w & -\epsilon I & \star & \star & \star \\ \Omega_{51} & 0 & 0 & 0 & 0 & -\epsilon I & \star & \star \\ \Omega_{61} & 0 & 0 & 0 & 0 & 0 & -\epsilon I & \star \\ 0 & \Omega_{61} & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{pmatrix} < 0 \qquad (24)$$

*where*

$$\begin{aligned} \Omega_{11} &= -\begin{pmatrix} r^2 X & \star \\ r^2 I & r^2 Y \end{pmatrix} \\ \Omega_{21} &= \begin{pmatrix} AX + \bar{B}C + \alpha X & A + \alpha I \\ \bar{A} + \alpha I & YA + B_u \bar{C} + \alpha Y \end{pmatrix} \\ \Omega_{22} &= -\begin{pmatrix} X & \star \\ I & Y \end{pmatrix}, \\ \Omega_{31} &= \sqrt{\alpha} \begin{pmatrix} CX & C \end{pmatrix}, \\ \Omega_{41} &= \sqrt{\alpha} \begin{pmatrix} F_w' & F_w' Y \end{pmatrix}, \\ \Omega_{51} &= \begin{pmatrix} EX + E_u \bar{C} & E \end{pmatrix}, \\ \Omega_{61} &= \epsilon \begin{pmatrix} H_1' & H_1' Y + H_2' \bar{B}' \end{pmatrix}. \end{aligned}$$

*As a result, a dynamic output feedback controller can be constructed as:*

$$\begin{aligned} A_K &= (N')^{-1}(\bar{A} - YAX - N' B_K CX - YB_u C_K M) M^{-1}, \\ B_K &= (N')^{-1}\bar{B}, C_K = \bar{C} M^{-1}. \end{aligned}$$

*where $X$ and $Y$ are arbitrary nonsingular matrices satisfying $M'N = I - XY$.*

**Proof:** (Necessity) It is obvious that the matrix $P$ and the controller parameters in the matrix $A_{cl}$ in (25) are unknown and occur in nonlinear fashion.

$$\begin{pmatrix} -r^2P & PA_{cl}' & \star & \star & \star & \star & \star & \star \\ A_{cl}P & -P & \star & \star & \star & \star & \star & \star \\ \sqrt{\alpha}C_{cl}P & 0 & -I & \star & \star & \star & \star & \star \\ \sqrt{\alpha}B_{cl}' & 0 & 0 & -\gamma^2 I & \star & \star & \star & \star \\ 0 & 0 & 0 & \sqrt{\alpha}E_w & -\epsilon I & \star & \star & \star \\ \tilde{E}P & 0 & 0 & 0 & 0 & -\epsilon I & \star & \star \\ \epsilon\bar{H}_1' & 0 & 0 & 0 & 0 & 0 & -\epsilon I & \star \\ 0 & \epsilon\bar{H}_1' & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{pmatrix} < 0 \quad (25)$$

Consequently, we apply a method of changing variables from Scherer [8] so that the matrix inequality (25) can be reduced to an LMI in all variables.

First, we define the partitioning of the matrix $P$ and $P^{-1}$ as

$$P := \begin{pmatrix} X & M' \\ M & U \end{pmatrix}, P^{-1} := \begin{pmatrix} Y & N' \\ N & Q \end{pmatrix},$$

where the order of controller $n_K$ is equal to the order of plant $n$. Then we pre-multiply and post-multiply (25) by $\mathcal{J}'$ and $\mathcal{J}$, respectively, where $\mathcal{J} = \text{diag}\{\Theta_2, \Theta_2, I, I, I, I, I, I\}$. Therefore, we can get:

$$\begin{pmatrix} \Theta_2'(-r^2P)\Theta_2 & \star & \star & \star & \star & \star & \star & \star \\ \Theta_2'A_{cl}P\Theta_2 & \Theta_2'(-P)\Theta_2 & \star & \star & \star & \star & \star & \star \\ \sqrt{\alpha}C_{cl}P\Theta_2 & 0 & -I & \star & \star & \star & \star & \star \\ \sqrt{\alpha}B_{cl}'\Theta_2 & 0 & 0 & -\gamma^2 I & \star & \star & \star & \star \\ 0 & 0 & 0 & \sqrt{\alpha}E_w & -\epsilon I & \star & \star & \star \\ \tilde{E}P\Theta_2 & 0 & 0 & 0 & 0 & -\epsilon I & \star & \star \\ \epsilon\bar{H}_1'\Theta_2 & 0 & 0 & 0 & 0 & 0 & -\epsilon I & \star \\ 0 & \epsilon\bar{H}_1'\Theta_2 & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{pmatrix} < 0$$

where

$$\Theta_1 := \begin{pmatrix} X & I \\ M & 0 \end{pmatrix}, \Theta_2 := \begin{pmatrix} I & Y \\ 0 & N \end{pmatrix}.$$

Then it is also apparent that

$$P\Theta_2 = \Theta_1, \Theta_2'P\Theta_2 = \Theta_1'\Theta_2 = \begin{pmatrix} X & \star \\ I & Y \end{pmatrix} > 0.$$

We substitute $A_{cl}, \bar{H}_1, \tilde{E}$, and $P$ in (25), and need to change the controller variables to new ones as:

$$\begin{aligned} \bar{A} &:= YAX + N'B_KCX + YB_uC_KM + N'A_KM \\ \bar{B} &:= N'B_K \\ \bar{C} &:= C_KM. \end{aligned}$$

Additionally, we can easily check each term in (25) as follows:

$$\begin{aligned} \Theta_2'(-r^2P)\Theta_2 &= -\begin{pmatrix} r^2X & \star \\ r^2I & r^2Y \end{pmatrix} \\ \Theta_2'A_{cl}P\Theta_2 &= \begin{pmatrix} AX + \bar{B}C + \alpha X & A + \alpha I \\ \bar{A} + \alpha I & YA + B_u\bar{C} + \alpha Y \end{pmatrix} \\ \Theta_2'P\Theta_2 &= \begin{pmatrix} X & \star \\ I & Y \end{pmatrix} \\ \sqrt{\alpha}C_{cl}P\Theta_2 &= \sqrt{\alpha}\begin{pmatrix} CX & C \end{pmatrix} \\ \sqrt{\alpha}B_{cl}'\Theta_2 &= \sqrt{\alpha}\begin{pmatrix} F_w' & F_w'Y \end{pmatrix} \\ \tilde{E}P\Theta_2 &= \begin{pmatrix} EX + E_u\bar{C} & E \end{pmatrix} \\ \epsilon\bar{H}_1'\Theta_2 &= \epsilon\begin{pmatrix} H_1' & H_1'Y + H_2'\bar{B}' \end{pmatrix} \end{aligned}$$

which imply that (24) holds.

(Sufficiency:) We assume that there exist $X, Y, \bar{A}, \bar{B}, \bar{C}$, and a scalar $\epsilon > 0$ satisfying (24). Now, assume $M$ and $N$ are arbitrary nonsingular matrices satisfying $I - XY = M'N$. Define

$$\begin{aligned} A_K &= (N')^{-1}(\bar{A} - YAX - \bar{B}CX - YB_u\bar{C})M^{-1} \\ B_K &= (N')^{-1}\bar{B} \\ C_K &= \bar{C}M^{-1} \\ P &= \Theta_2\Theta_1^{-1} \end{aligned}$$

where

$$\Theta_1 = \begin{pmatrix} X & I \\ M & 0 \end{pmatrix}, \Theta_2 = \begin{pmatrix} I & Y \\ 0 & N \end{pmatrix}.$$

Note that $P = \Theta_2\Theta_1^{-1} = P' = (\Theta_1')^{-1}\Theta_2'$. Next, we will show that the above $P$ is a solution to (25). Substituting $A_K, B_K, C_K$, and $P$ into (25) to yield:

$$\begin{aligned} -r^2P &= (\Theta_2')^{-1}\Omega_{11}\Theta_2^{-1},\ A_{cl}P = (\Theta_2')^{-1}\Omega_{21}(\Theta_2)^{-1} \\ -P &= (\Theta_2')^{-1}\Omega_{22}\Theta_2^{-1},\ \sqrt{\alpha}C_{cl}P = \Omega_{31}\Theta_2^{-1} \\ \sqrt{\alpha}B_{cl}' &= \Omega_{41}\Theta_2^{-1},\ \tilde{E}P = \Omega_{51}\Theta_2^{-1},\ \epsilon\bar{H}_1' = \Omega_{61}\Theta_2^{-1}. \end{aligned}$$

where $\Omega_{11}, \ldots, \Omega_{61}$ are the same as (24). ■

**Remark 2** *It is obvious that (24) is not an LMI due to the product of a scalar $\epsilon$ with variables $Y$ and $\tilde{B}$, respectively. As a result, the LMI software fails to solve (24). However, we are able to achieve difficulty by setting $\epsilon$ as a prior value and then apply the LMI software. We need to tune $\epsilon$ until the solver returns a feasible solution.*

**Remark 3** *In this problem, when the $\mathcal{H}_\infty$ constraint is not considered ($\gamma \to \infty$), the problem reduces to robust controller design with pole-clustering constraints considered in [5]. Hence, (13) is reduced to the following LMI:*

$$\begin{pmatrix} -r^2P & \star & \star & \star \\ A_\alpha P & -P & \star & \star \\ EP & 0 & -\epsilon I & \star \\ 0 & \epsilon H_1' & 0 & -\epsilon I \end{pmatrix} = \begin{pmatrix} -r^2P & \star & \star & \star \\ A_\alpha P & -P & \star & \star \\ EP & 0 & -I & \star \\ 0 & H_1' & 0 & -I \end{pmatrix} < 0$$

*Similarly, when the pole-clustering constraint is not included, the problem is reduced to the standard robust $\mathcal{H}_\infty$ controller design. As a result, (13) is reduced to the following LMI:*

$$\begin{pmatrix} A'P + PA & \star & \star & \star & \star \\ CP & -I & \star & \star & \star \\ F_w' & 0 & -\gamma^2 I & \star & \star \\ EP & 0 & E_w & -\epsilon I & \star \\ \epsilon H_1' & 0 & 0 & 0 & -\epsilon I \end{pmatrix} < 0 \qquad (26)$$

**Remark 4** *It is easy to design the state feedback controller as in [6] when all state variables can be available.*

## 4 Simulation Results

To demonstrate the efficiency of the proposed method, we first study the robustness of the proposed controller against the variation of system parameters. By using LMI Control Toolbox [2], the proposed robust $\mathcal{H}_\infty$ controller is compared with the standard robust $\mathcal{H}_\infty$ one from (26).

From a system model in (1) as follows:

$$\tilde{A} = \begin{pmatrix} -a & b & 0 & 0 \\ 0 & -c & c & 0 \\ -d & 0 & -e & -e \\ 0.6 & 0 & 0 & 0 \end{pmatrix}, \tilde{F}_w = \begin{pmatrix} -b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\tilde{B}_u = \begin{pmatrix} 0 & 0 & e & 0 \end{pmatrix}', C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

The range of the system parameter variation are given below:

| $a$ | $b$ | $c$ |
|---|---|---|
| [0.033, 0.1] | [4,12] | [2.564,4.762] |
| $d$ | $e$ | |
| [9.615,17.857] | [3.081,10.639] | |

Here, we need to employ important parameters as follows:

- Nominal parameters

$$A = \begin{pmatrix} -0.0665 & 8 & 0 & 0 \\ 0 & -3.663 & 3.663 & 0 \\ -6.86 & 0 & -13.736 & -13.736 \\ 0.6 & 0 & 0 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & 0 & 13.736 & 0 \end{pmatrix}'$$
$$F = \begin{pmatrix} -8 & 0 & 0 & 0 \end{pmatrix}'$$

- The structure of considered uncertainties in (3).

$$H_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{pmatrix}', E_u = \begin{pmatrix} 0 \\ 0 \\ 0.4121 \end{pmatrix},$$
$$E = \begin{pmatrix} -0.0168 & 0 & -0.3779 \\ 2 & -0.5495 & 0 \\ 0 & 0.5495 & -0.4121 \\ 0 & 0 & -0.4121 \end{pmatrix}',$$
$$E_w = \begin{pmatrix} 2 & 0 & 0 \end{pmatrix}'.$$

We would like to enforce all closed-loop poles in $\mathcal{D}(80.28, 80)$, according to Remark 2, assume $\epsilon = 0.00101$, and $\gamma = 582.88$ with circular pole constraints, while $\gamma = 165.05$ without circular ones. By using a unit step input in this simulation, the responses of $y(t)$ are shown in Figure 2 where consist of the upper and lower bounds of parameter variations and nominal parameter which are defined in Table 1.

Table 1: Parameters in three cases

| Case | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| Upper case | 0.1 | 12 | 4.762 | 17.857 | 10.630 |
| Nominal case | 0.0665 | 8 | 3.663 | 13.736 | 6.86 |
| Lower case | 0.033 | 4 | 2.564 | 9.615 | 3.081 |

In addition, it is well-known that the difference in the shape of the transient responses from the nominal responses are an indication of the performance robustness. To show performance robustness of all method, we define the norms ($\mathcal{N}$) and absolute sums ($\Sigma$) as follows:

$$\mathcal{N}_{ubp} = \left[\sum_{j=0}^{k}[x_{np}(t_k) - x_{ubp}(t_k)]^2\right]^{1/2}$$
$$\mathcal{N}_{lbp} = \left[\sum_{j=0}^{k}[x_{np}(t_k) - x_{lbp}(t_k)]^2\right]^{1/2}$$
$$\Sigma_{ubp} = \sum_{j=0}^{k}|x_{np}(t_k) - x_{ubp}(t_k)|$$
$$\Sigma_{lbp} = \sum_{j=0}^{k}|x_{np}(t_k) - x_{lbp}(t_k)|$$

where $x_{np}(t_k)$, $x_{ubp}(t_k)$ and $x_{lbp}(t_k)$ represent the nominal, upper bound and lower bound value of any state $x(t_k)$ at time $t_k$, respectively. The numerical results of the performance robustness are shown in Table 2.

Table 2: Comparison of performance robustness of both methods for the time responses

| Method | $\mathcal{N}_{upb}$ | $\mathcal{N}_{lpb}$ | $\Sigma_{upb}$ | $\Sigma_{lpb}$ |
|---|---|---|---|---|
| Our method | 0.0521 | 0.1566 | 3.5469 | 11.687 |
| The standard method | 0.0864 | 0.2572 | 7.6624 | 23.111 |

From Figure 2 and Table 2, it is obvious that the deviation from the nominal value of the perturbed system (upper and lower limits) is much less in case of the proposed method compared with that of two methods mentioned. This means that the performance robustness of the proposed controller is superior compared to these methods. Also, it is very clear that the transient performance is improved and can be achieved such as maximum overshoot, settling time, and rise time which are both smaller and shorter, while the other methods cannot. These are related to pole locations in a specified region. In Figure 3, it is apparent that all closed-loop poles are placed in a desired region. From these figures, it is obvious that the transient responses in increment frequency deviation $y(t)$ does decay faster and exhibit smaller overshoot as well as shorter settling time, when the the circular pole constraint is included in our proposed controller design. Roughly speaking, the more all closed-loop poles is pushed toward LHP, the more the settling time and overshoot decrease quickly. Also, we are able to use this circular pole constraint in regulating the control input and avoiding the saturation in any systems.

## 5 Conclusions

In this paper, a controller design problem involving both pole-clustering constraint and the prespecified $\mathcal{H}_\infty$ norm constraint using LMI approach is considered. The aim of this paper is to design a suitable controller which is able to achieve robust stability and to improve the transient responses of the uncertain system, simultanuously. The necessary and sufficient conditions for the existence of a desired controller have proved and the feasible solution to LMI can be used to find a desired controller. In comparision with the standard robust $\mathcal{H}_\infty$ controller, the system responses with the proposed controller gives better transient responses with respect to settling time, rise time, and maximum overshoot in all cases of the system parameter variations.

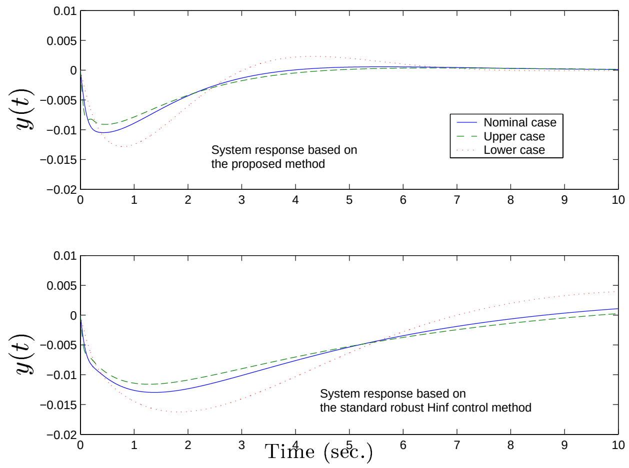


Figure 2: The responses of $y(t)$ from two methods

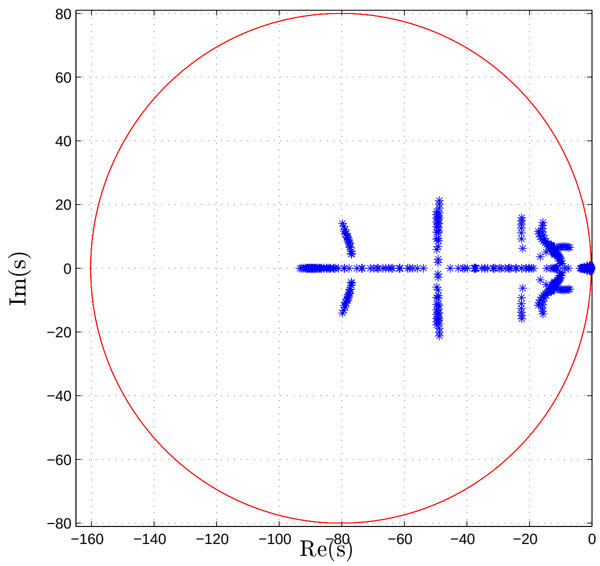


Figure 3: Pole locations

Besides, it is shown that the transient responses of the proposed controller with pole-clustering constraints is improved.

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## List of Symbols

- $\Re$: The set of real numbers.
- $\Re^m$: The set of real $m$-vector.
- $\Re^{m\times n}$: The vector space of $m \times n$ real matrices.
- $X'$: The transpose of a matrix $X \in \Re^{m\times n}$.
- $I_m$: The identity matrix of size $m$ of the identity of linear operator. We omit the subscript when $m$ can be determined from context.
- $X^{-1}$: The inverse of $X$ or the inverse of linear operator $X$, *i.e.*, $XX^{-1} = I$.
- $X > 0$: The symmetric $X$ is positive definite, *i.e.*, $X = X^T$ and $z^T X z > 0$ for all $z \in \Re^n$.
- $\in$: belongs to
- $\star$: The symbol $\star$ is used for terms that are induced by symmetry whenever symmetric block matrices or long matrix expressions are encountered, i.e., $\begin{pmatrix} A & \star \\ B & C \end{pmatrix} = \begin{pmatrix} A & B' \\ B & C \end{pmatrix}$.
- $\blacksquare$: The square block $\blacksquare$ signals an end to theorem statements and proofs.