

Stabilizing of an Inverted Pendulum Using Computed Feedback Linearization Technique

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Abstract

In the paper, a stabilization controller for balanced rod inverted pendulum is proposed. The proposed technique, based on pole placement over the first order linearized model around a trajectory, approximately linearize the pendulum system. Once the system is approximately linear, it becomes asymptotically stable. Several tools such as roots and phase portraits are used to analyze the nonlinear system behavior. The simulation and experiment, based on ECP 505 inverted pendulum plant, is used to demonstrate and verify the proposed technique. Compared to the LQG controller, the proposed technique shows a superior result, i.e., both the transient response and steady state error are improved and the region of stability is also wider.

Keywords: Inverted pendulum, Feedback linearization, Under-actuated mechanical system.

1. Introduction

The inverted pendulum has been widely used to investigate and develop new control strategies that can effectively deal with nonlinearities. The main challenge is that the plant is a nonlinear, under-actuated mechanical system with unstable zero dynamics and is to be controlled such that the position is at its unstable equilibrium. There are a number of research efforts to stabilize this system since the late 1970s. The most popular technique to stabilize this system at a desired position is using an optimal linear quadratic regulator (LQR) based upon a linearized plant.

The other candidate for nonlinear controller is based on Artificial Neural Network and Fuzzy logic controller [1,2,3]. The state feedback gains, as a nonlinear function of state, can be realized using neuro-fuzzy system. In recent years, the sliding mode controller based on fuzzy logic [4] draws much attention in control the nonlinear system such as an inverted pendulum.

There are a number of configurations of inverted pendulum systems such as the classic “cart and pole” inverted pendulum [5], the double- [6] and triple-inverted pendulum on a cart, the rotary inverted pendulum [1,4,7], parallel type inverted pendulum [2], the spherical inverted pendulum [8], two link inverted pendulum [9], etc. All of these systems are nonlinear, under-actuated system with unstable zero dynamics.

Since an inverted pendulum is under-actuated, the system is not completely linearizable. Thus, the control task is normally separated into two different tasks, first a swing up control and a latter a balance control. In order to

swing up the pendulum, several strategies based on partial feedback linearization, passivity and energy shaping are proposed [5]. When the system is brought close enough to the equilibrium, the control strategy switches to a stabilizing control. Since, the system is linearizable around the equilibrium, the LQR based controller could be design. The paper aims to address how to systematically deal with under-actuated nonlinear mechanical system using the balanced rod inverted pendulum as an example. Several tools have been used to analyze and design the optimal controller for such system.

The proposed technique called “computed feedback linearization” is based on pole placement over the first order linearized model around a trajectory. The strategy is to keep the closed loop system’s roots fixed, thus the system becomes approximately linear. If the roots are on the left half plane, i.e., their real part is less than zero, the system is asymptotically stable. Both simulation and experiment based on ECP 505 balanced rod inverted pendulum are used to demonstrate and verify the technique.

The paper is organized into seven sections. The first section is introduction. The second is the model of the balanced rod invert pendulum. The third and fourth sections are about the LQG and computed feedback controller respectively. Simulation and experimental results are shown and discussed in the fifth and sixth sections respectively. The seventh section gives the conclusion of the paper.

2. Model of the ECP 505 Inverted Pendulum

The ECP 505 inverted pendulum consists of a pendulum rod which supports the sliding balance rod. The DC servo motor, below the pendulum rod, is used to drive the sliding balance rod through a drive shaft, a pulley and a belt. This sliding rod is to be steered horizontally in order to control the vertical pendulum rod. The center of gravity, and thus the system dynamics, can be altered by adjusting the brass counter weight position. The position of the sliding rod and the pendulum rod are sensed by two encoders, one at the back of the motor and the other one at the pivoting base of the pendulum.

The mathematical model of the ECP 505 system can be derived using Euler-Lagrange equation or the Newtonian approach. However, the first approach is used in this paper as follows;

The Lagrange equation is:

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}} \right] - \frac{\partial L}{\partial q} = Q_q \quad (1)$$

where

$L = T - V$

T : kinetic energy

V : potential energy

Q_q : generalized forces

q : generalized coordinates

The q is selected as $[\theta, x]^T$. Thus, the kinetic energy, T , is

$$T = \frac{1}{2} J_0 (\dot{x})^2 + \frac{1}{2} m_1 \dot{x}^2 + m_1 l_0 \dot{x} \dot{\theta}$$

where

$$J_0(x) = J_1 + J_2 + m_1(l_0^2 + x^2) + m_2 l_c^2$$

The potential energy, V , is

$$V = m_1 g(l_0 \cos(\theta) - x \sin(\theta)) + m_2 g l_c \cos(\theta)$$

The Euler-Lagrange equations result in

$$m_1 \ddot{x} + m_1 l_0 \ddot{\theta} - m_1 x \dot{\theta}^2 - m_1 g \sin(\theta) = F(t)$$

$$m_1 l_0 \ddot{x} + J_0 \ddot{\theta} + 2m_1 x \dot{\theta} - (m_1 l_0 + m_2 l_c) g \sin(\theta) - m_1 g x \cos(\theta) = 0$$

This system is highly nonlinear, having two degrees of freedom with only one actuator. There is also a nonlinear coupling between the actuated and the un-actuated degrees of freedom.

Table 1 Plant Parameter

Symbol	Value	Description
m_1	0.213 kg	Mass of sliding rod
m_2	1.785 kg	Mass of complete assembly minus m_1
J_0	0.036 kg.m ²	The equivalent J of the system
l_0	0.330 m	Length of pendulum rod
l_c	0.0281 m	The position of center of m_2
g	9.81 m/s ²	Gravity

The system parameters used in the simulation are given in table 1. This is the nominal parameters for the ECP 505 plant and the controller is design based on these values.

3. LQ Controller Design based upon linearized plant Local Linearization about equilibrium

A linearized approximation of the system about the equilibrium point $[x_e \theta_e] = [0 \ 0]$ which only the first two (zeroth and first order) terms of Taylor's series expansion are used is found as

$$m_1 \ddot{x} + m_1 l_0 \ddot{\theta} - m_1 g \theta = F(t)$$

$$m_1 l_0 \ddot{x} + J_0 \ddot{\theta} - (m_1 l_0 + m_2 l_c) g \theta - m_1 g x = 0$$

where

$$J_{0e} = J_1 + J_2 + m_1 l_0^2 + m_2 l_c^2$$

The result is the same as setting $\sin(\theta)$ and $\cos(\theta)$ equal to θ and 1 respectively and the $\dot{x}$ and $\dot{\theta}$ are setting equal to zero.

The linearized approximation can be written in state space form as

$$\dot{x} = Ax + BF(t) \quad (2)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{m_2 l_c g}{J^*} & 0 & \frac{m_1 g}{J^*} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(J^* - m_2 l_0 l_c) g}{J^*} & 0 & \frac{-m_1 l_0 g}{J^*} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ l_0 \\ -\frac{J^*}{J^*} \\ 0 \\ \frac{J_{0e}}{m_1 J^*} \\ m_1 J^* \end{bmatrix}$$

Where $x = [\theta \ \dot{\theta} \ x \ \dot{x}]^T$

$$J^* = [J_{0e} - m_1 l_0^2]$$

LQG Controller

Since the open loop system is both naturally unstable and non-minimum phase, the full state feedback is recommended to control the system. The LQR synthesis, via matrix Riccati equation, is used to determine the optimal gains which the following cost function is minimized

$$J = \int (x' Q x + R u^2) dt \quad (3)$$

Subject to the linear time invariant dynamics

$$\dot{x} = Ax + Bu \quad (4)$$

The matrices Q and R are positive semi-definite and definite respectively.

Once the optimal gains is determined, the feedback control law is

$$u(t) = -Kx(t) \quad (5)$$

In the simulation, the Q and R are set to 1 and 10 respectively. This results in the optimal gains $K = [4.6314 \ 1.4197 \ 8.8969 \ 2.6202]$. The roots of the closed loop system are the eigenvalues of $[A - BK]$ and is found to be

$$r_{1,2} = -0.2998 \pm 6.2053i, \ r_{3,4} = -3.3318 \pm 0.1275i$$

4. Computed Feedback Linearization

The LQG controller is based upon the linearized plant, thus, the performance is well predictable only at the neighborhood of the equilibrium point. When the system travels far away from the equilibrium, the control performance degrades or the system may become unstable.

Consider the following nonlinear system.

$$\dot{x}(t) = f(x(t), u(t)) \quad (6)$$

where $x(t) \in \mathbb{R}^N$ is a state vector at time t .

$u(t) \in \mathbb{R}^M$ is an input vector.

The linearized model along the trajectory at state x_0 can be written as

$$\dot{x} = A_0 x + f(x_0, u_0) - A_0 x_0 + B u \quad (7)$$

where A_0 is the Jacobian $\frac{\partial f(x_0(t), u_0(t))}{\partial x}$

B is an $N \times M$ matrix $\frac{\partial f(x_0(t), u_0(t))}{\partial u}$

Set the full state feedback $u(x_0, x)$ as

$$u = \frac{(A_0 x_0 - f(x_0, u_0))}{B} - Kx \quad (8)$$

Thus, the closed loop system becomes

$$\dot{x} = (A_0 - BK)x \quad (9)$$

The feedback gains, K , can be determined such that the eigenvalues of $(A_0 - BK)$ is at specified location using pole placement technique.

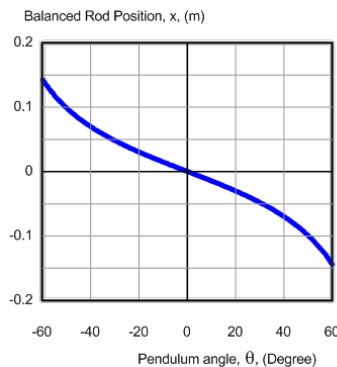


Figure 1 The sub-equilibrium function

The proposed technique, called computed feedback linearization, determines the appropriate gains at every position such that the system roots are fixed at a specified location. First, the gains at the equilibrium point are determined by the LQG synthesis as explained earlier. Once the gains at the equilibrium are determined, the system roots at this point are obtained. The gains at other system positions are determined based upon the linearized plant around a trajectory at each time step. Then, the pole placement technique is used to determine the appropriate gains such that the closed loop system having the roots fixed at the specified location. Once the roots are fixed and the nonlinear feedforward term, $f(x, e)$ is zero out, the closed loop system becomes approximately linear. Since all roots are in the left half

plane, the exponential stable is obtained. Actually, this technique determines the nonlinear function of $u(x, \theta)$ to gives the closed loop system linear.

Since the system is not only nonlinear but also under-actuated, the control problem becomes much more difficult. We are unable to determine the nonlinear function $u(x, \theta)$ at every position of θ and x in order to transform the system into a linear one. In the other word, the system is not completely linearizable. The state is thus partitioned into actuated degree, x , and under-actuated degree, θ . In our strategy, the $x(\theta)$, the sub-equilibrium function, is determined first. This sub-equilibrium function determines the position of x that gives the position of θ in equilibrium. Generally speaking, some degrees of freedom are in equilibrium at these points, while the rests are not. Since only the θ is in equilibrium but the x may not be, thus, the control force, u , is required to control the x position. The nonlinear system is linearizable at the sub-equilibrium positions and can be controlled such that the roots are fixed at a specified location. It is noted that if the whole system is stable, the system will be moving around these positions. Thus, the linearized model is accurate enough and the closed loop system can be approximately linear. The linear control theory can be used to justify the control performance of the closed loop system.

In this paper, the system starts somewhere near the sub-equilibrium, the system then asymptotically converts to a desired sub-equilibrium position. If the system is at point far from the sub-equilibrium, another control strategy should be used to bring the system close to the sub-equilibrium. The strategy to bring the system close to the sub-equilibrium is typically based on partial feedback linearization, passivity and energy shaping, and is not mentioned in this paper.

5. Simulation Results

In this section, the matlab® simulation based on 'ode45' is used to demonstrate the technique. In Fig. 2, the step response is asymptotically stabilized at ± 0.35 radian when using the proposed controller. The LQG controller is unable to stabilize the plant at this position. The tracking response, shown in Fig. 3, demonstrate that the system is able to track the sinusoidal trajectory quite well. In both cases, the linear behavior is observed.

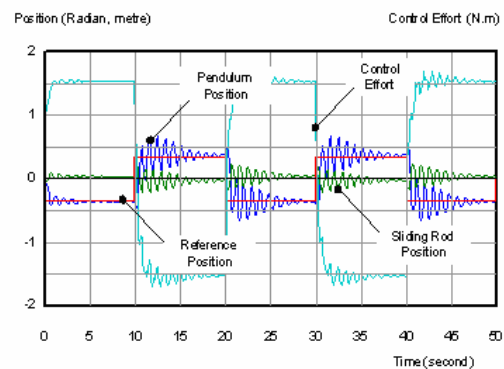


Figure 2 Stabilizing the pendulum at 0.35 radian

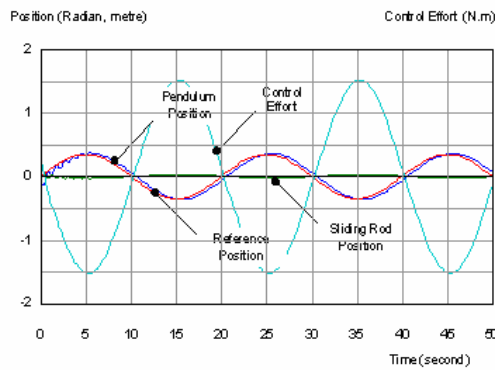
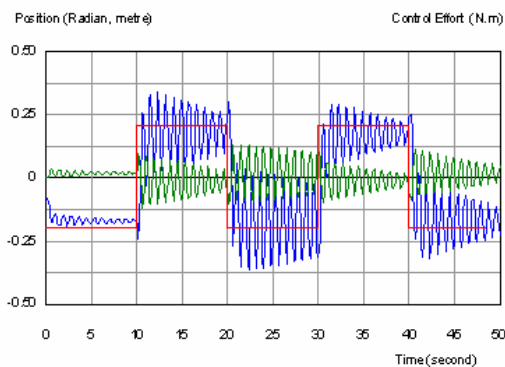
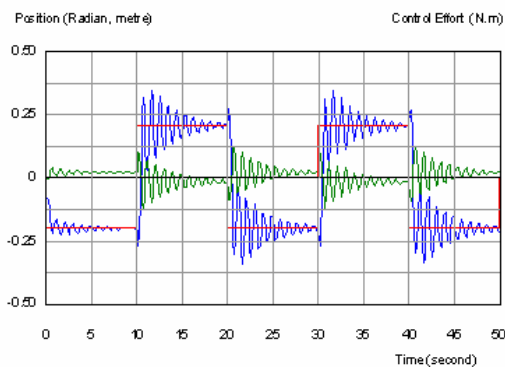


Figure 3 Tracking Performance at 0.7 pp Sine, 0.05 Hz Reference Trajectory.

Compared to the LQG controller, the proposed technique shows superior performance in both transient response and steady state error as shown in Fig. 4. The feedback gain designed by LQG synthesis at equilibrium is difficult to stabilize the system at the position far from the equilibrium (0.2 radian in the case shown in Fig. 4). The steady state error is also detected. The proposed technique accurately computed the gain at each time step, such that the roots are fixed and the feedforward term is cancelled out. Thus, the control performance is significantly improved.



a) LQG Controller

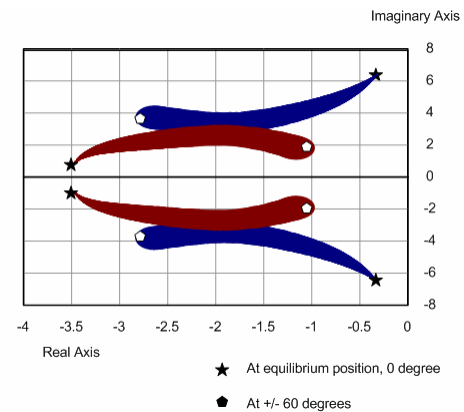


b) Computed Feedback Linearization

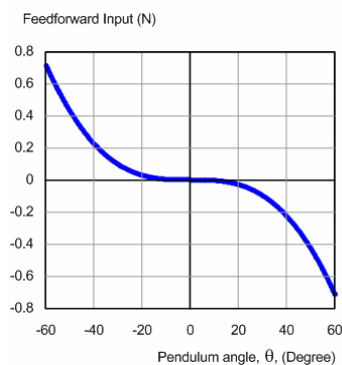
Figure 4 Stabilizing Performance at 0.2 Radian compared to LQG controller

The roots portrait in Fig. 5 demonstrates system characteristic in s -domain when constant feedback gain is used to stabilize the system. The gain is optimally

design at the equilibrium based on LQR synthesis. When the system travels away from this point, but still in the sub-equilibrium, the closed loop poles drifts as shown in the plot. The width of the root trajectory shows the nonlinear feedforward input which increasingly arises as the system travels away from the equilibrium. This term can be regards as a predetermined disturbance and can be cancelled out to enhance the control performance. In the pendulum example, all of the system roots are in the left haft plane and actually the nonlinear feedforward input is the one that makes the system unstable.



a) The roots trajectory



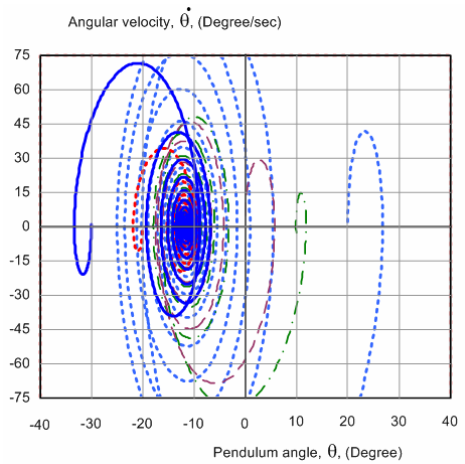
b) The remaining feedforward input

Figure 5 The roots portrait

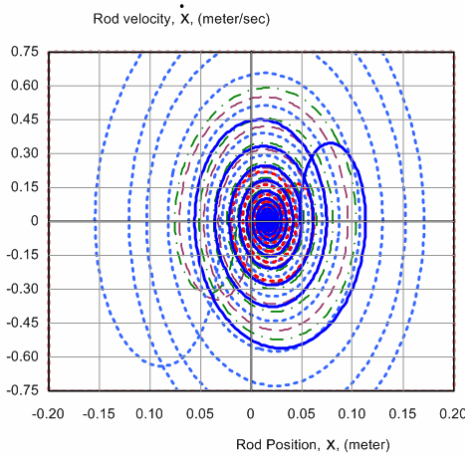
In Fig. 5b, the feedforward input is plotted against the pendulum rod angle. The magnitude of feedforward term is low and it can be negligible when rod angle is within $\pm 20^\circ$. The ECP 505 plant has a stopper at these positions to limit the rod motion beyond this range. The LQG can not stabilize the plant at the position beyond this range as demonstrated earlier while the proposed controller eliminates this feedforward input and thus significantly improves the system stability.

Table 2 Starting positions

Case	θ	x	$\dot{\theta}$	$\dot{x}$
1	-30	0.0480	1	0.01
2	-20	0.0303	-2	0.02
3	0	0	5	0.05
4	10	-0.0146	-3	-0.03
5	20	-0.0303	2	0.01



a) Pendulum angle



b) Balanced Rod Position
Figure 6 The phase portrait

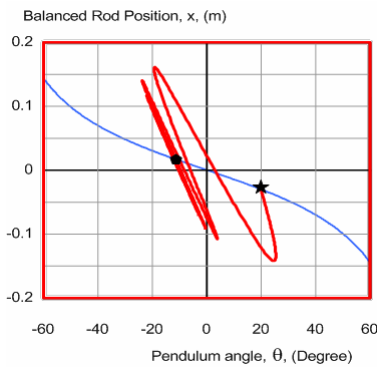


Figure 7 The x - θ trajectory of case #5

The proposed technique is also tested with various starting positions shown in Table 2. The phase diagrams of both pendulum rod and balanced rod are shown in Fig. 6. The system asymptotically converted to the goal at 0.2 radian in every case. For the case number #5, the trajectory is shown in Fig. 7. This shows that the system travels around the sub-equilibrium position along the way to the goal.

6. Experimental Results

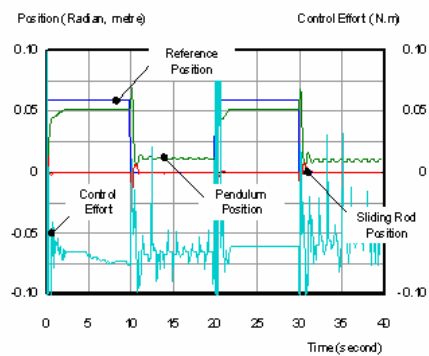
The ECP 505 inverted pendulum, shown in Fig 8, is used to validate the proposed technique. The control

algorithm is implemented through Delta-tau Pmac lite DSP card that comes with the plant. The sampling rate is set to 0.00884 second and digital filter is also implemented.

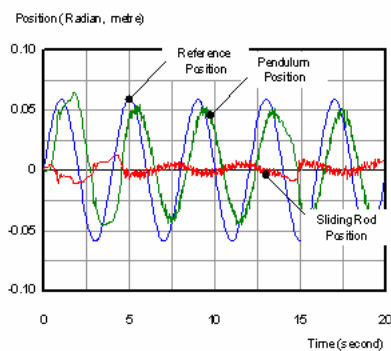


Figure 8 The ECP 505 plant

Since the plant is designed to limit the range of the pendulum rod to $\pm 20^\circ$ where the nonlinear feedforward input can be negligible, the LQG controller is able to stabilize the system. Thus, the superiority of the proposed technique is not well demonstrated since both controllers give almost the same results.



a) Step response



b) Sinusoidal Reference at 0.25 Hz
Figure 9 Experimental Result

The step response, shown in Fig. 9a, demonstrate the stability of the proposed technique. However, the steady state error is observed. This is because the controller is design based on imprecise model and could be further improved. Fig. 9b shows that the plant can track the sinusoidal trajectory quite well. We are now modifying the plant and also its controller for further investigation.

7. Conclusion

Although, the under-actuated inverted pendulum cannot be completely linearized, it can be accurately linearized around the sub-equilibrium. Thus, the LQG controller could be designed based upon the linearized model around this point, i.e., the controller gains are computed at each discrete time step. The gains are determined such that the closed loop poles are fixed in the s-plane and thus the system becomes approximately linear. The paper also address the nonlinear feedforward term which arises when the system is far from the equilibrium and thus should be cancelled out to significantly improve the system stability. The simulation results demonstrate that our technique is superior than the LQG based upon the linearized model around the equilibrium. The experiment is also used to validate the technique.

References

- [1] S. Liu, L. Cui, J. Chen, "Research of Rotary Inverted Pendulum Using Fuzzy Strategy Based on Dynamic Query Table," The 4th World Congress on Intelligent Control and Automation, Shanghai, P.R.China, June 2002.
- [2] J. Yi, N. Yubazaki, K. Hirota, "Stabilization Fuzzy Control of Parallel-Type Double Inverted Pendulum System," The 9th IEEE international conference on fuzzy systems, May 2000.
- [3] T.S. Leng, N.S. Leng, "Control of an Inverted Pendulum using a Neuro-Fuzzy Controller," Asia-Pacific Workshop on Advances in Motion Control, July 1993.
- [4] C.G. Lhee, J.S. Park, H.S. Ahn, D.H. Kim, "Sliding Mode-like Fuzzy Logic Control with Seft-Turning the Dead Zone Parameters," IEEE trans. Fuzzy Syst., Vol. 9, No. 2, pp. 343-348, April 2001.
- [5] W. Zhong, H. Rock, "Energy and Passivity Based Control of the Double Inverted Pendulum on a Cart," The 2001 IEEE international conference on control applications, Mexico city, Mexico, September 2001.
- [6] J. Rubi, A. Rubio, A. Avello, "Swing-up Control Problem for Self Electing Double Inverted Pendulum," IEE proc. Control Theory Appl. Vol. 149, No 2, pp.169-175, March 2002.
- [7] B. Potsaid, J.T. Wen, "Optimal Mechanical Design of a Rotary Inverted Pendulum," The 2002 IEEE/RSJ international conference on intelligent robots and systems EPFL, Lausanne, Switzerland, Oct 2002.
- [8] X. Albouy, L. Praly, "On the Use of Dynamic Invariants and Forwarding for Swinging Up a Spherical Inverted Pendulum," The 39th IEEE conference on decision and control, Sydney, Australia, December 2000.
- [9] K. Chrisman, "An Alternative Inverted Pendulum Apparatus for Education," The American control conference, Seattle, Washington, USA, June 1995.