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2D Potential Flow Analysis Using Nodeless Finite Element Method

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Abstract

Nodeless finite element method is presented to predict the low–speed and inviscid flow behavior. The paper first describes 2D potential flow theory. Finite element formulations based on Bubnov–Galerkin method and nodeless triangular element, the computational procedure and its boundary conditions are then represented. The validated examples of the proposed technique are flow pass a circular cylinder problem and flow pass an elliptical cylinder problem. The nodeless finite element results are compared to the solutions using quadratic triangular element to assess the efficiency of nodeless finite element method.

Introduction

Potential flow is an inviscid flow of low speed flow. There are many applications that potential theory can predict the nature of flow problem such as flow in pipe, seepage flow, free surface flow, and etc. There are several numerical techniques applied to predict potential flow behavior. Finite element method is one of those methods, that is applied to solve potential flow problem for years. For 2D potential flow problem, there are several element types such as linear triangular element, quadratic triangular element, quadrilateral element, and etc. are commonly used for finite element calculation.

This paper presents nodeless finite element method for predicting the potential flow behavior. Nodeless triangular element and its element interpolation function are described. Then, the computational procedure and its boundary conditions are shown. Next, the computational nodeless finite element solutions are validated with the exact solution and finite element solutions with quadratic triangular element.

2. Theory

2.1 Governing differential equation

Potential flow problem is governed by Laplace's equation as shown below [1, 2].

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (1)$$

where ϕ is the velocity potential. The relation of potential function and velocity is in the following formulation.

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} \qquad (2)$$

2.2 Element interpolation function and finite element matrices [3 – 6]

Nodeless triangular element consists of 3 nodes and 3 nodeless per element as shown in figure 1.

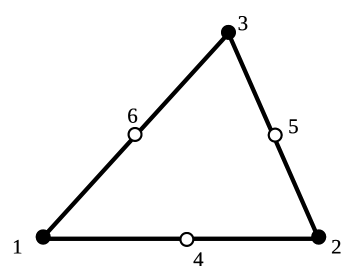


Figure 1. Nodeless element and its connectivity.

Its element interpolation functions are in the form,

$$N_i(x,y) = \frac{1}{2A}\left(a_i + b_i x + c_i y\right) \qquad i = 1, 3 \qquad (3)$$

$$N_4 = 4N_1 N_2 \quad ; \quad N_5 = 4N_2 N_3 \quad ; \quad N_6 = 4N_1 N_3 \qquad (4)$$

where

$$a_i = x_j y_k - x_k y_j \ , \ b_i = y_j - y_k \ ,$$
$$c_i = x_k - x_j \qquad\qquad i, j=1,3$$

a_i , b_i , and c_i coefficients are obtained by cyclically permuting the subscripts, and A is triangular area.

After applying MWR in equation 1, the finite element equation based on Bubnov–Galerkin method is obtained to solve 2D potential flow problem.

$$[K]\{\phi\} = \{Q\} \tag{5}$$

where

$$[K] = \int_A [B]^T [B] dA$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \end{bmatrix}$$

$$\frac{\partial N_1}{\partial x} = \frac{y_2 - y_3}{2A} \quad , \quad \frac{\partial N_1}{\partial y} = \frac{x_3 - x_2}{2A}$$
$$\frac{\partial N_2}{\partial x} = \frac{y_2 - y_1}{2A} \quad , \quad \frac{\partial N_2}{\partial y} = \frac{x_1 - x_3}{2A}$$
$$\frac{\partial N_3}{\partial x} = \frac{y_1 - y_2}{2A} \quad , \quad \frac{\partial N_3}{\partial y} = \frac{x_2 - x_1}{2A}$$
$$\frac{\partial N_4}{\partial x} = \frac{(y_2 - y_3)N_2 + 2(y_3 - y_1)N_1}{A}$$
$$\frac{\partial N_4}{\partial y} = \frac{(x_3 - x_2)N_2 + 2(x_1 - x_3)N_1}{A}$$
$$\frac{\partial N_5}{\partial x} = \frac{(y_3 - y_1)N_3 + 2(y_1 - y_2)N_2}{A}$$
$$\frac{\partial N_5}{\partial y} = \frac{(x_1 - x_3)N_3 + 2(x_2 - x_1)N_2}{A}$$
$$\frac{\partial N_6}{\partial x} = \frac{(y_1 - y_2)N_1 + 2(y_2 - y_3)N_3}{A}$$
$$\frac{\partial N_6}{\partial y} = \frac{(x_2 - x_1)N_1 + 2(x_3 - x_2)N_3}{A}$$

$$\{Q\} = \int_S \left(\frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y \right) \{N\} dS$$

Pressure can be computed by using Bernoulli's equation.

$$\frac{p}{\rho} + \frac{V^2}{2} + gZ = \text{Constant} \tag{6}$$

where V = total velocity

3. Examples

In this paper, nodeless finite element method is developed. There are two potential flow problems selected to validate the proposed method. First problem is the flow passes a cylinder. The solution is compared with the exact solution and computational solution using quadratic triangular element. The second problem is the flow passes an elliptical cylinder problem.

3.1 Flow passes a cylinder

Flow enters from the left of cylinder with velocity U_∞ = 1.0. There is a cylinder obstacle flow at the center as shown in figure 2.

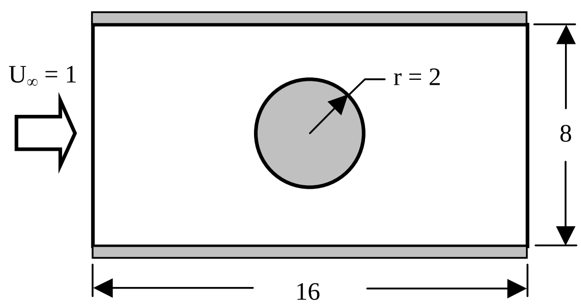


Figure 2 – Flow past a cylinder problem statement.

From the velocity boundary condition at an inlet and a wall, it can be transformed to be the potential boundary condition. Normal potential gradient at the inlet is minus inlet velocity and zero at every wall as shown in figure 3. Finite element model consists of 145 nodes and 242 elements.

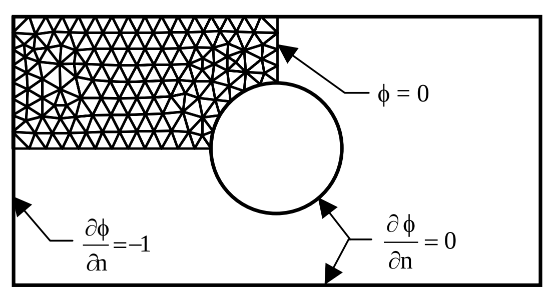


Figure 3 – Finite element model and boundary condition.

Nodeless finite element is applied to compute the potential, velocity and pressure solution. Potential solution on the cylinder wall is compared with the exact solution and computational solution using linear and quadratic triangular element as shown in figure 4.

The exact solution is in the form,

$$\phi = 2U_\infty \cos\theta \tag{7}$$

Nodeless and quadratic triangular element have the same accuracy and its error is 1.2% less than the linear triangular element.

Solution of potential and stream function are plotted in figure 5. Figure 6 shows the u–velocity and pressure distribution. The red color shows maximum value and pink value represents minimum value.

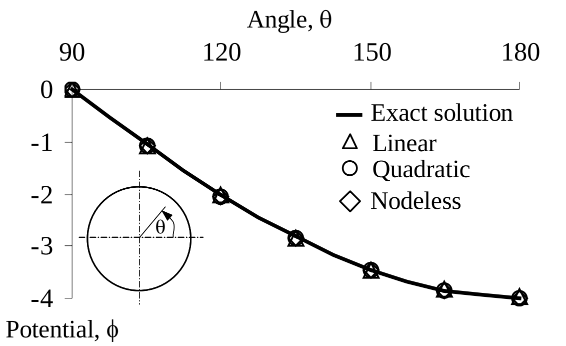
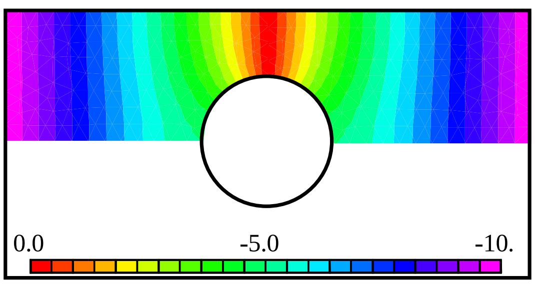
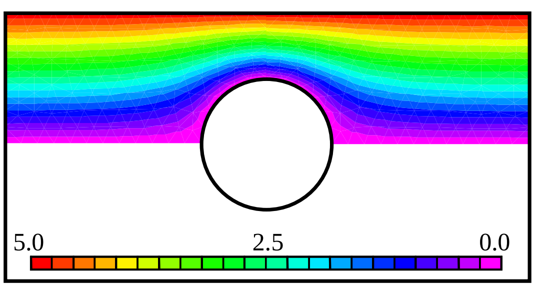


Figure 4 – Plot of potential function and cylinder angle.

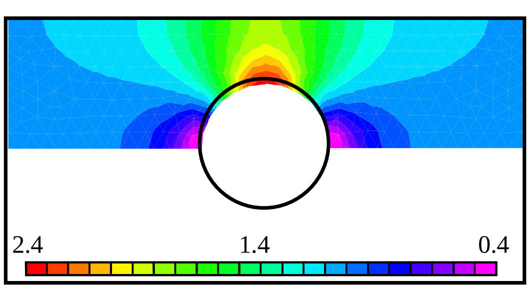


(a) Potential contours

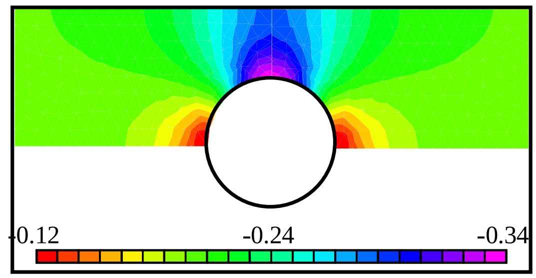


(b) Stream contours

Figure 5 – Potential and stream contours.



(a) u–Velocity contours



(b) Pressure contours

Figure 6 – Pressure and u–velocity contours.

3.2 Flow past an inclined ramp

Flow $U_\infty = 1$ entering from the left past an inclined ramp (45 degree) as shown in figure 7. After flow past an inclined ramp it will occur compression at this corner and pressure will be maximum at this compression corner. Then, flow will change direction again after past the second corner. Flow will be in horizontal direction again. At this second corner the expansion flow will happen and pressure is minimum.

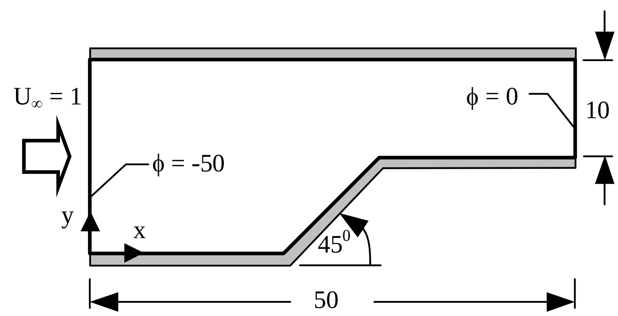


Figure 7 Flow past an inclined ramp problem statement.

The finite element model consists of 1,071 nodes and 2,000 elements as depicted in figure 8.

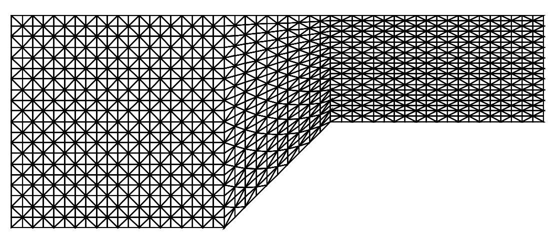


Figure 8 – Finite element model.

After solving the problem with nodeless finite element method, it obtains potential distribution as shown in figure 9, streamlines in figure 10, u–velocity distribution in figure 11, and pressure distribution in figure 12.

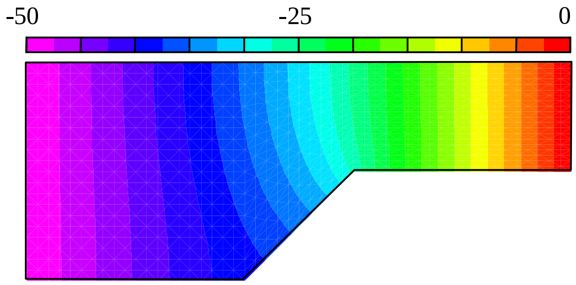


Figure 9 – Potential contours of flow past an inclined ramp

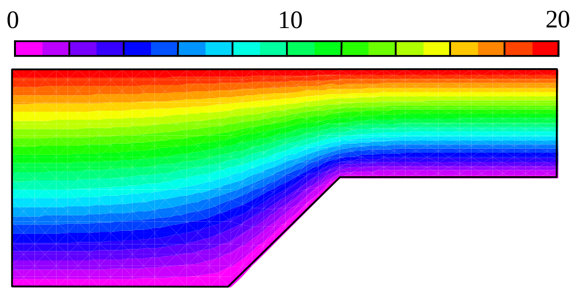


Figure 10 – Streamlines of flow past an inclined ramp

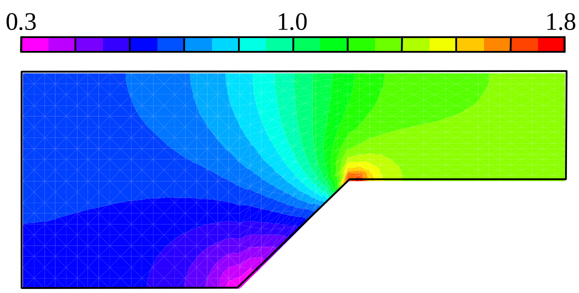


Figure 11 – u – velocity distribution of flow past an inclined ramp

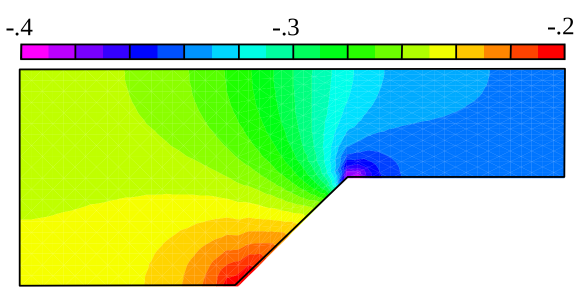


Figure 12 – Pressure distribution of flow past an inclined ramp

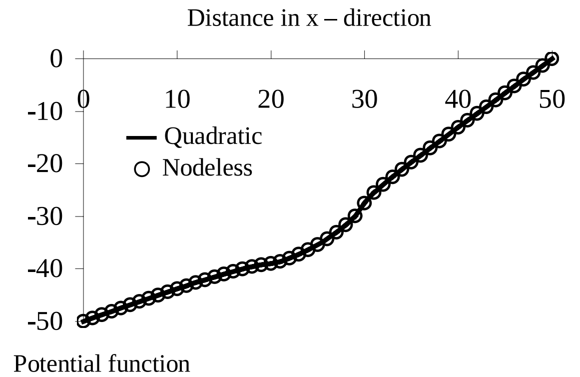


Figure 13 – Plot potential function along x – direction.

Figure 13 shows the potential solution from nodeless and quadratic triangular element. They have a good agreement.

4. Conclusion

This paper presents nodeless finite element method to solve 2D potential flow problem. The element interpolation and finite element equation are shown. The proposed method is validated with two examples, flow past a cylinder and flow past an inclined ramp. The accuracy is compared with the exact solution and finite element solution using linear and quadratic triangular elements. The results of two examples show that the accuracy of nodeless finite element method is better than linear element and equal quadratic triangular element.

5. Acknowledgment

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6. References

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