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A Study on Three Finite Element Schemes for Transient Convection Dominated Flows

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Abstract

A study on three finite element schemes, the Streamline Upwind Petrov-Galerkin method, the Taylor-Galerkin method and the Streamline Upwinding Finite Element method, for convection dominated flow is presented to identify the most suitable scheme for the analysis of transient viscous incompressible flow. These schemes were examined with three pure convection problems which are the pure convection of a cone in constant velocity field, the mixing of a hot with a cold front problem, and the Smith & Hutton test case. It has been found that the Streamline Upwind Petrov-Galerkin method exhibits minimal dissipation error among the three methods.

1. Introduction

Transient viscous incompressible flow is an important part of the fluid-structure interaction problem which can be found in many engineering applications such as wind-induced vibration of a bridge, flow in an internal combustion engine, and pulsation of the blood flow through an artery in biomechanical engineering. The difficulties in the transient viscous incompressible flow analysis are the non-linear phenomenon of the convection term and the time discretization. Various finite element methods have been proposed to analyze such problem. One of the most successful techniques proposed for finite element method is the Streamline Upwind Petrov-Galerkin method [1] which modifies the weighting functions for using with the Galerkin's method.

Another method is the Taylor-Galerkin method [2] which uses the Taylor series expansions including the second- and third-order terms for marching the solution in time. The procedure yields a generalized governing equation which is discretized in time only with the spatial variable being left continuous. Such equation is successively discretized in space using the convectional Bubnov-Galerkin finite element method.

Another successful method is the Streamline Upwinding Finite Element method [3]. The method evaluates convection terms directly along the local streamline instead of modifying the weighting functions and then, the Crank-Nicolson method is implemented for time discretization.

This paper presents a comparative study of the three finite element schemes namely, the Streamline Upwind Petrov-Galerkin method, the Taylor-Galerkin method and the Streamline Upwinding Finite Element method for transient convection dominated flow. Numerical solutions are obtained for three pure convection problems which are the pure convection of a cone in constant velocity field, the mixing of a hot with a cold front problem and the Smith & Hutton test case.

2. Finite Element Algorithms

2.1 Governing Equation

The following governing equation for pure convection is considered herein to examine the performance of the finite element schemes in discretizing the convection term.

$$\frac{\partial \phi}{\partial t} + U_j \frac{\partial \phi}{\partial x_j} = 0 \qquad (1)$$

where ϕ is a quantity being transported by the flow field, t is the time, U_j is the velocity components with j = 1, 2.

2.2 Streamline Upwind Petrov Galerkin Method (SUPG)

To derive the finite element equations, the three-node triangular element is used in this study. The element assumes linear interpolation as,

$$\phi(x,y) = N_i\,\phi_i \tag{2}$$

where i = 1, 2, 3 and N_i is the element interpolation functions. Then, the method of weighted residuals is applied to the governing equation (1) with the modified weighting function (3) excluding the time derivative $\partial\phi/\partial t$ to which the standard Galerkin weighting of N_i is attached,

$$N_i + \frac{\Delta t}{2} U_j \frac{\partial N_i}{\partial x_j} \tag{3}$$

The procedure leads to the discretized finite element equations in the form,

$$[M]\{\phi^{n+1} - \phi^n\} = -\Delta t\,[C + K_u]\{\phi^n\} \tag{4}$$

where

$$[M] = \int_\Omega \{N\}\lfloor N \rfloor d\Omega \tag{5a}$$

$$[C] = \int_\Omega \{N\}\left\lfloor U_j \frac{\partial N}{\partial x_j} \right\rfloor d\Omega \tag{5b}$$

$$[K_u] = \frac{\Delta t}{2}\int_\Omega \left\{U_j \frac{\partial N}{\partial x_j}\right\}\left\lfloor U_j \frac{\partial N}{\partial x_j} \right\rfloor d\Omega \tag{5c}$$

where Ω is the element area, Δt is the time increment, the superscript n is the time level. These element equations are assembled to yield the global equations. After that, nodal boundary conditions and initial conditions are imposed prior to solving for the transport property, ϕ, at each time step.

2.3 Taylor Galerkin Method (TG)

To develop discretized finite element equations by the Taylor-Galerkin method, a forward-time Taylor series expansion is employed by including the second- and third-time derivatives to the time derivative term of governing equation (1) as follows,

$$\frac{\partial\phi}{\partial t} = \frac{\phi^{n+1} - \phi^n}{\Delta t} + \frac{\Delta t}{2}\frac{\partial^2\phi}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3\phi}{\partial t^3} \tag{6}$$

By successive differentiations of equation (1),

$$\frac{\partial^2\phi}{\partial t^2} = \left(U_j \frac{\partial}{\partial x_j}\right)\left(U_j \frac{\partial\phi}{\partial x_j}\right) \tag{7}$$

and

$$\frac{\partial^3\phi}{\partial t^3} = \left(U_j \frac{\partial}{\partial x_j}\right)\left(U_j \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial t}\right)\right) \tag{8}$$

and then, combine equations (6-8) with equation (1) to yield,

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = -\left(U_j \frac{\partial\phi}{\partial x_j}\right) - \frac{\Delta t}{2}\left(U_j \frac{\partial}{\partial x_j}\right)\left(U_j \frac{\partial\phi}{\partial x_j}\right) - \frac{\Delta t^2}{6}\left(U_j \frac{\partial}{\partial x_j}\right)\left(U_j \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial t}\right)\right) \tag{9}$$

The last term in equation (9) originates from the third time derivative term in the Taylor series expansion. After substitution $\partial\phi/\partial t$ by $(\phi^{n+1} - \phi^n)/\Delta t$, such term is transferred to the left-hand side of the equation. Then, by applying method of weighted residuals with the standard Galerkin weighting function, the finite element equations are obtained in the form,

$$[K]\{\phi^{n+1} - \phi^n\} = \{R\} \tag{10}$$

where

$$[K] = \int_\Omega \left(\{N\}\lfloor N \rfloor + \frac{\Delta t^2}{6}\left\{U_i \frac{\partial N}{\partial x_i}\right\}\left\lfloor U_i \frac{\partial N}{\partial x_i} \right\rfloor\right) d\Omega \tag{11}$$

$$\{R\} = -\int_\Omega \left(\{N\} + \frac{\Delta t}{2}\left\{U_i \frac{\partial N}{\partial x_i}\right\}\right) d\Omega \left(U_i \frac{\partial\phi^n}{\partial x_i}\right) \tag{12}$$

The above element equations from the Taylor-Galerkin method are assembled to form the global equations, nodal boundary conditions and initial conditions are applied prior to solving the system of equations.

2.4 Streamline Upwinding Finite Element Method (SUFE)

For the streamline upwinding finite element formulation, a special treatment for the convection terms is incorporated. These terms are approximated by a monotone streamline upwinding formulation to be used with the triangular element [4]. In this approach, the convection term is first written in the streamline coordinates as,

$$U_s \frac{\partial\phi}{\partial s} \tag{13}$$

where U_s and $\partial/\partial s$ are the velocity and the gradient along the streamline direction, respectively. These terms are evaluated by a streamline tracing method which keeps track of the direction of the flow within the element.

Using the standard Galerkin approach, the governing equation is multiplied by weighting function, N_i, to yield the element equations in the form,

$$[M]\{\partial\phi/\partial t\} + [A]\{\phi\} = 0 \tag{14}$$

where the coefficient matrix $[A]$ contains the known contributions from the convection term and the matrix $[M]$ is similar as shown in Eq. (5a). For time discretization, the recurrence relations [5] is applied by the substitution of $\partial\phi/\partial t$ with $(\phi^{n+1}-\phi^{n})/\Delta t$ and let $\{\phi\} = (1-\theta)\{\phi^{n}\}+\theta\{\phi^{n+1}\}$, where θ = 0.5 for Crank-Nicolson method. Then, the finite element equations (14) become,

$$\left(\frac{1}{\Delta t}[M]+\theta[A]\right)\{\phi^{n+1}\} = \left(\frac{1}{\Delta t}[M]-(1-\theta)[A]\right)\{\phi^{n}\} \quad (15)$$

where the $\{\phi^{n+1}\}$ on the left-hand side of the equation are unknowns, and all of the terms on right-hand side are known.

3. Results and Discussion

In order to examine and compare the ability of the above finite element schemes, the three test case problems are presented in this section.

3.1 The Pure Convection of a Cone in Constant Velocity Field

This test case is chosen to evaluate the ability of each finite element scheme in capturing steep gradients of a cone which advected in a constant velocity field. The computational domain is a rectangular region as illustrated in Fig. 1 and the initial shape of a cone is defined by Eq. (16) and is shown in Fig. 2.

$$\phi(x,y) = 0.75\exp\left[-2\frac{(x-0.5)^2+(y-0.5)^2}{0.06}\right] \quad (16)$$

The velocity field is specified to be uniform over the entire domain and is given by

$$u(x,y) = 1 \quad (17a)$$

$$v(x,y) = 0 \quad (17b)$$

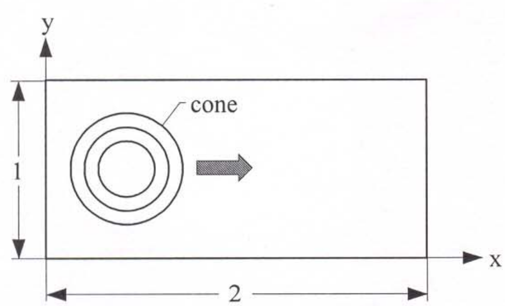


Fig. 1 - Problem statement of the pure convection of a cone.

The time step of 0.01 is used in this study. To ensure that the cone moves past completely the entire domain, the total number of time steps equal to 200 is selected. Figure 3 shows the exact solutions of a moving cone in constant velocity field at each time step.

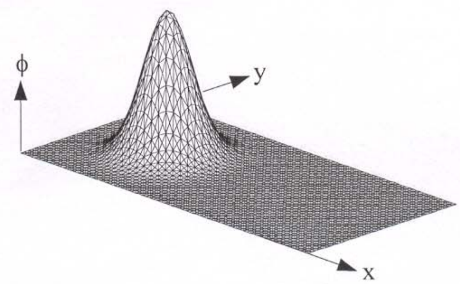


Fig. 2 - Initial shape of a cone in constant velocity field.

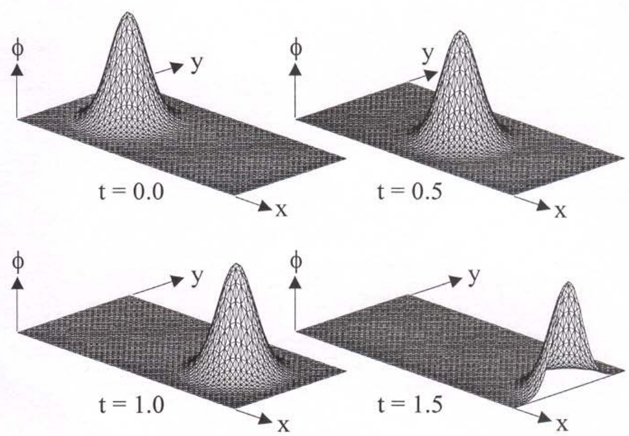


Fig. 3 - Exact solutions of a cone in constant velocity field.

To evaluate the efficiency of the studied schemes, the predicted heights of the cone at t = 1.5 are compared with the height of the exact solution as shown in Table 1. The table indicates that the height of the cone from the Streamline Upwind Petrov-Galerkin method and the Taylor-Galerkin method was found 99.7 percent of its original value, indicating the minimum dissipation error.

Table 1 - Compare the height of a cone at t = 1.5 of the studied schemes with the exact solution.

	The height of a cone	Error (%)
Exact	0.750	-
SUPG	0.748	0.3
TG	0.748	0.3
SUFE	0.451	39.9

3.2 The Mixing of A Hot with a Cold Front Problem

The second test case is used to study a narrow region of high temperature gradient from top to bottom called a "front" which is then twisted by a steady vortex similar to that observed on daily weather maps [6-7]. The computational domain is the square region and is defined by $-4 \leq x, y \leq 4$ as shown in Fig. 4. The tangential velocity of the vortex is given as,

$$V_T(r) = \frac{\tanh(r)\ \cosh^{-2}(r)}{0.3849} \tag{18}$$

where V_T is the tangential velocity of the vortex and r is the distance from the origin of the coordinate system. For the Cartesian coordinates, the velocity components can by expressed as,

$$u(x,y) = -\frac{y}{r} V_T(r) \tag{19a}$$

$$v(x,y) = \frac{x}{r} V_T(r) \tag{19b}$$

In this case, the analytical solution [8] of the governing equation (1) is,

$$\phi(x,y,t) = -\tanh\left(\frac{y}{2}\cos\left(\frac{V_T}{r}t\right) - \frac{x}{2}\sin\left(\frac{V_T}{r}t\right)\right) \tag{20}$$

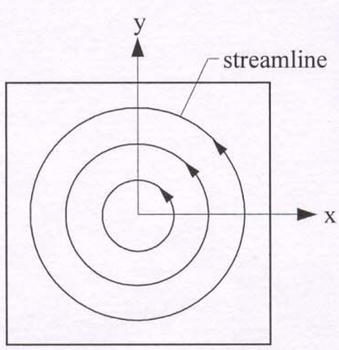


Fig. 4 - Computational domain and streamline contours of a hot with a cold front in steady vortex problem.

The initial condition, calculated from Eq. (20) at t = 0, is shown in Fig. 5. The numerical solution contours, obtained after the front has twisted for 4 time units or t = 4, and the exact solution at this time are shown in Fig. 6. To evaluate the accuracy of the studied schemes, the relative errors of Eq. (21) are calculated and illustrated in Table 2.

$$\varepsilon = \frac{\sqrt{\sum |\phi - \phi_{exact}|^2}}{n} \tag{21}$$

where ε is the relative error and n is the number of node.

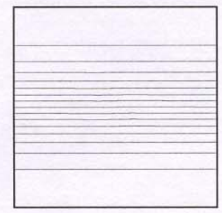


Fig. 5 - Contour plots of the initial condition for the mixing of a hot with a cold front problem.

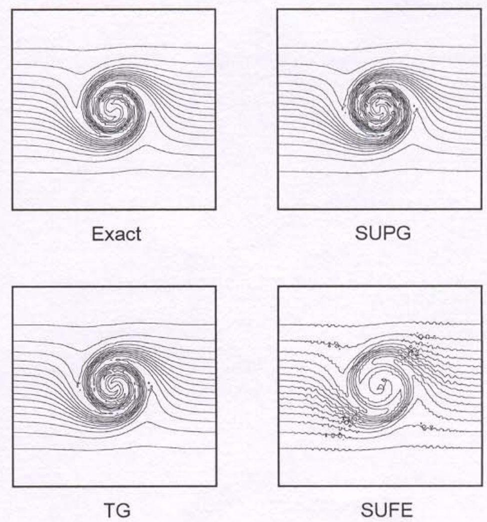


Fig. 6 - Contour plots of the numerical solutions and exact solution.

Table 2 - Relative error of the studied schemes at t = 4.

	Relative error, ε
SUPG	1.746×10^{-4}
TG	1.673×10^{-4}
SUFE	8.841×10^{-4}

Table 2 shows that Taylor-Galerkin method can provide the lowest relative error compare to the other methods and the contour plots is in very good agreement with the exact analytical solution.

3.3 Smith & Hutton Test Case

The last test case is first presented by Smith and Hutton [9]. The geometry for this problem is the rectangular domain as shown in Fig. 6. The velocity field is given by the following relations,

$$u(x,y) = 2y\left(1-x^2\right) \tag{22a}$$

$$v(x,y) = -2x\left(1-y^2\right) \tag{22b}$$

This velocity field produces the pattern of streamlines depicted in Fig. 7. With the exception of the outlet part of the boundary, ϕ is specified on the boundary as,

$$\phi = 1 + \tanh\left[10(2x+1)\right] \qquad y = 0,\ -1 \le x \le 0 \tag{23a}$$

$$\phi = 1 - \tanh 10 \begin{cases} x = -1 & 0 \le y \le 1 \\ y = \ \ 1 & -1 \le x \le 1 \\ x = \ \ 1 & 0 \le y \le 1 \end{cases} \tag{23b}$$

The value of ϕ is essentially 0 on $x = \pm 1$ and $y = 1$ and is closed to 2 at the origin of coordinates. The climb from 0 to 2 occurs very sharply halfway along the inlet side.

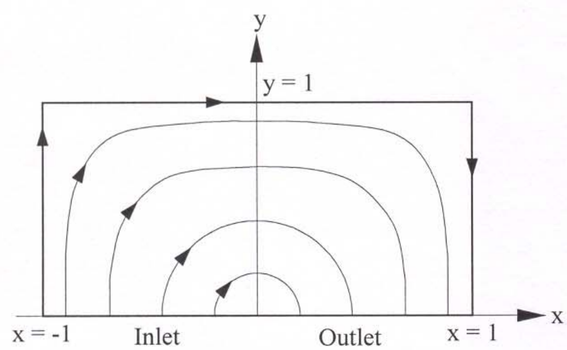


Fig. 7 - Geometry and streamline contours for Smith and Hutton test case.

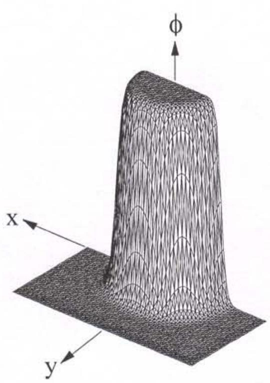
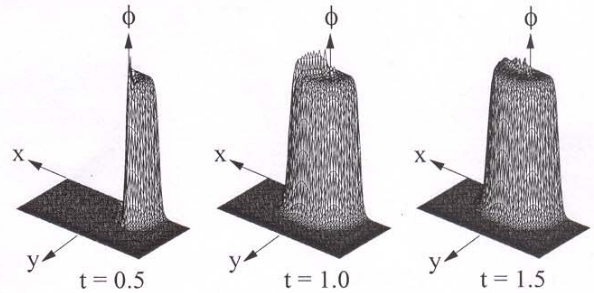
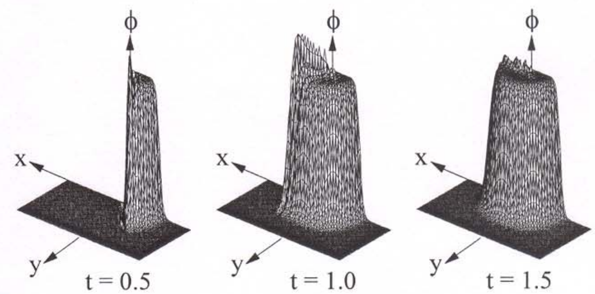


Fig. 8 - Distribution of ϕ in pure convection at steady state.

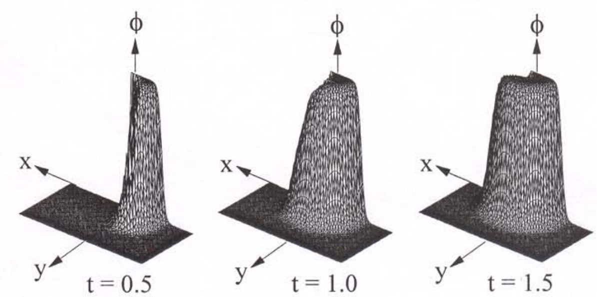
For pure convection problem, the inlet profile should propagate to the exit plane without any diffusion as shown in Fig. 8. Figures 9a-c show the propagation of ϕ of each finite element scheme before it reaches the steady state. The figures indicate that the Streamline Upwind Petrov-Galerkin method and the Taylor-Galerkin method provide sharp discontinuity but with non-physical spatial oscillations. The Streamline Upwinding Finite Element method, however, does not generate any oscillations.



(a) Streamline Upwind Petrov Galerkin method.



(b) Taylor Galerkin method.



(c) Streamline Upwinding Finite Element method.

Fig. 9 - Distribution of ϕ of three finite element schemes at each time step.

To evaluate the performance of the algorithms in capturing the steep gradient, the profiles of ϕ along the outlet boundary of the domain from each algorithm are compared as shown in Fig. 10. The Figure shows that, at steady-state condition, the Streamline Upwind Petrov-Galerkin method exhibits the best performance in capturing such gradient while the solution from Streamline Upwinding Finite Element method produces the least accurate result.

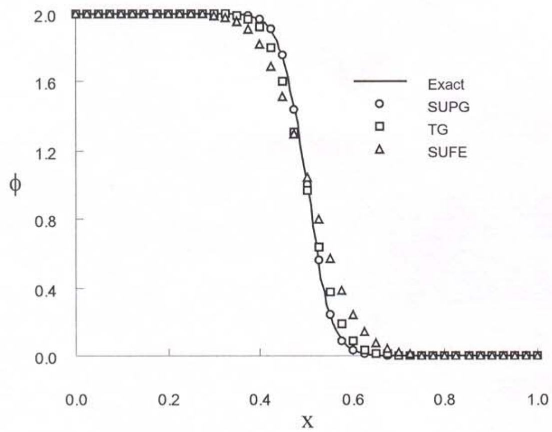


Fig. 10 - Comparison of the distributions of ϕ along the outlet boundary.

4. Conclusions

This paper presents a study of the three finite element formulations which are the Streamline Upwind Petrov-Galerkin method, the Taylor-Galerkin method and the Streamline Upwinding Finite Element method for analysis of transient convection dominated flow. The discretization of each formulation and its finite element equations are described. The capabilities of the finite element algorithms have been evaluated by three pure convection test case problems. The computational results show that, from the point of view of the solution accuracy, the Streamline Upwind Petrov-Galerkin can capture steep gradient accurately with slight spatial oscillations.

5. Acknowledgement

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