# Analytical Kinematics and Dynamics for 5 axis H-4 parallel Manipulator 

Kummun Chooprasird and Viboon Sangveraphunsiri*<br>Robotics and Manufacturing Lab.<br>Department of Mechanical Engineering,Faculty of Engineering, Chulalongkorn University Bangkok 10330, Thailand.<br>*E-mail: viboon.s@eng.chula.ac.th


#### Abstract

This work presents analysis and design of an unique hybrid 5 degree-of-freedom robotic manipulator based on a H-4 Family of Parallel Mechanisms with three degree-of-freedom in translational movements and one degree-of-freedom in rotational movement (orientation angle) at the tool tip of the arm together with another degree-offreedom coming from a single axis rotating table. This manipulator can be used in a rapid prototype application for cutting soft materials. Forward or direct kinematics, inverse kinematics and Jacobian of the purposed configuration are derived in detail as well as equations of motion of the manipulator arm. The equations of motion or the dynamic model are derived from Lagrangian formulation and are shown to be suitable in real-time feedback controls. The accuracy of the kinematics, forward and inverse, Jacobian, and the dynamic model derived in this work are assured by comparing the results obtained from using MATLAB developed in this work with the result from the ADAMS solver with the manipulator arm 3D solid model data. The comparisons between the two numerical results are very promising.


Keywords: H-4 Parallel Robot, Resolve acceleration, Lagrangian formulation

## 1. Introduction

Many advantages of the parallel mechanism robots over serial mechanism, such as structural stiffness, load capacity, accurate precision, speed and acceleration performance, induce the parallel manipulator to the topics of research activities in the academic community for several years since the 6-DOF parallel mechanism for flight simulator proposed by Gough and Stewart. Although the disabilities due to the parallel configurations are the small working volume and small movement for orientation to reach a workpiece, the benefits of these mechanisms have still been taking in development continuously in fields of robotic. Recently, many applications utilize the parallel mechanisms in industries such as high-speed laser machine using three RUU parallel chains (RUU: Revolute-UniversalUniversal) by Delta Group, 3-DOF (RUU chains) the FlexPicker IRB 340 for high-speed picking and packing
applications by ABB, 6-DOF "Hexapod" as consisted of 6 UPS parallel chains (UPS: Universal-PrismaticSpherical).

As known before, the minimum requirement to give maximum flexibility in between tool and workpiece orientation is five degrees of freedom. It means that the tool can be oriented in any angle relative to workpiece. In the case where a part with complex surfaces, 3-DOF manipulator cannot perform good enough surface cutting. Or event worst, 3-DOF manipulator can not fulfill all the surfaces of the part. This is not the case for 5 -DOF manipulator. In present time, the manufacturing industries are widely known the 5-DOF in parallel structure than before. Besides the advantages mentioned above, the motion characteristics are also preferable such as the errors in the arms are averaged instead of build up as in series structure, the load is distributed to all arms, low inertia due to actuators are mount on a fixed base. The purposed configuration is good for a rapid prototype application in the form of a master-slave application with force control algorithms.

Due to multiple close-chain links make the dynamic analysis of parallel manipulators more complicated than serial manipulators. And dynamic analysis of various types of parallel mechanisms has been studied by many researchers. Do and Yang, 1988; Guglielmetti and Longchamp, 1994; Tsai and Kohli, 1990 formulated dynamic models from Newton-Euler method. While Lebret 1993; Miller and Clavel, 1992; Miller and Clavel, 1992; Pang and Shahingpoor, 1994, used Lagrange Equations method. And virtual work method is studied by Codourey and Burdet, 1997; Miller, 1995; Tsai,1998; Wang and Gosselin, 1997; Zhang and Song, 1993. To obtain the dynamic model by using Newton-Euler formulation, the equations of motion of each body need to be written which leads to a large number of equations and consumes large computation time when used in a real-time control system. Unnecessary computation of reaction forces as done in the Newton-Euler formulation are eliminated in the Lagrangian formulation. Additional coordinates along with a set of Lagrangian and some model simplification make the Lagrange formulation more efficient than Newton-Euler formulation. In this paper we derive the dynamic model of the $\mathrm{H}-4$ parallel
robot by using the Lagrangian. Sangveraphunsiri has been working on the kinematics analysis for sometimes as shown in [6], [8] and [9], some of the works here are based on those works.

The intention of this paper is to design and evaluate the kinematics of a 5-DOF manipulator based on the H-4 family parallel structure together with one rotational table as shown in Figure 1. Lagrange formulation has been introduced to analyze dynamics of H-4 manipulator. The computer simulation using the commercial software (ADAMS solver), with the solid model data of the system, is carried out to compare the results from the dynamic model derived. From the simulation results, it is shown that the complexity and the performance of this kind of configuration are suitable for complex surface machining application with or without cutting force control. Tele-operation with force reflecting control can also be implemented with this configuration.

## 2. Design Consideration

The desired H-4 parallel robot must have at least 4DOF which will produce 3 translation motions and 1 rotary motion. In order to have large rotational movement in $y$-axis, the mechanism will consist of two mechanical chains connect to two separated platforms. Each chain has two degrees of freedom moving along the x -axis of the platform as shown in Figure 1 and Figure 2. The work volume is depends on the 4 parameters as the length of limb, the size of platform, 2 c , the different between machine frame and platform, ( $\mathrm{b}-\mathrm{a}$ ), and the offset between two separated platform, 2d. The 4 variables ( $l_{1}$, $1_{2}, l_{3}$, and $l_{4}$ ) are used for specifying the position in ( $\mathrm{x}, \mathrm{y}$, $z$ ) and rotation in $y$-axis of the tool tip.


Figure 1. Proposed design

## 3. Kinematics Modeling

In this section, we derive the relationships between the joint variables of the 5 -DOF of the mechanism, $\mathbf{q}=\left[\begin{array}{lllll}l_{1} & l_{2} & l_{3} & l_{4} & \alpha\end{array}\right]$ and working coordinate system $\mathbf{x}=\left[\begin{array}{llllll}x_{w} & y_{w} & z_{w} & I_{w} & J_{w} & K_{w}\end{array}\right]$ as shown in Figure 1
and Figure 2, where $I_{w}, J_{w}$, and $K_{w}$ are the directional cosine of the working frame $\left[\begin{array}{lll}x_{w} & y_{\omega} & z_{w}\end{array}\right]$.

## Inverse Kinematics

Typically, working paths will be defined in Cartesian space or the working coordinate system. Inverse kinematics is used for obtaining the joint commands from the specified working paths.


Figure 2. Robot joint configuration
From Figure 2, vector $\mathbf{C}_{1} \mathbf{B}_{1}$ and $\mathbf{C}_{1} \mathbf{B}_{2}$ are symmetry about $\mathbf{Z}_{\mathbf{c}} \mathbf{Y}_{\mathbf{c}}$-plane if $\theta=0$ and vector $\mathbf{O}_{\mathbf{c}} \mathbf{C}_{\mathbf{1}}$ and $\mathbf{O}_{\mathbf{c}} \mathbf{C}_{\mathbf{3}}$ point to opposite direction. So, inverse position relationships can be obtained as follow:

$$
l_{1,2}=x_{a}+w \pm \sqrt{R^{2}-\left(y_{a}-e\right)^{2}-\left(z_{a}-g e\right)^{2}}
$$

From geometrical restrictions, we can conclude that

$$
\begin{align*}
& l_{1}=x_{a}+w+\sqrt{R^{2}-\left(y_{a}-e\right)^{2}-\left(z_{a}-g e\right)^{2}}  \tag{1}\\
& l_{2}=x_{a}-w-\sqrt{R^{2}-\left(y_{a}-e\right)^{2}-\left(z_{a}-g e\right)^{2}} \tag{2}
\end{align*}
$$

The coordinate $\left(x_{a}, y_{a}, z_{a}\right)$ represents the vector position of point $\mathbf{C}_{\mathbf{1}}$ with respect to origin of the link $l_{1}$ or $l_{2}$. The relationship between tool-tip coordinate frame, $\{x, y, z\}$, and center coordinate frame, $\left\{x_{c}, y_{c}, z_{c}\right\}$, can be derived as following

$$
\begin{align*}
x= & x_{c}+\left(T_{L}+c\right) \times \sin (-q) \\
= & x_{a}+\left(T_{L}+2 c\right) \times \sin (-q) \\
y= & -\left(y_{a}+b\right)+a \\
& (1 b) \\
-z= & z_{a}+\left(T_{L}+2 c\right) \times \cos (-q)-d \tag{1c}
\end{align*}
$$

Similarly, the others actuator-end effecter position relationship become,

$$
\begin{align*}
& l_{3}=x_{b}+w+\sqrt{R^{2}-\left(y_{b}-e\right)^{2}-\left(z_{b}-g e\right)^{2}}  \tag{3}\\
& l_{4}=x_{b}-w-\sqrt{R^{2}-\left(y_{b}-e\right)^{2}-\left(z_{b}-g e\right)^{2}} \tag{4}
\end{align*}
$$

where $\left(x_{b}, y_{b}, z_{b}\right)$ is coordinate of the point $\mathbf{C}_{3}$ with respect to origin of the link $l_{3}$ or $l_{4}$ as shown in Figure 2 and it can be shown that

$$
\begin{align*}
x & =x_{c}+\left(T_{L}+c\right) \sin (-q)=x_{b}+T_{L} \times \sin (-q)  \tag{2a}\\
y & =y_{c}=\left(y_{b}+b\right)-a  \tag{2b}\\
-z & =z_{b}+d+T_{L} \cos (-q) \tag{2c}
\end{align*}
$$

From equation (3), (4), and (2a)-(2c), $x_{b}, y_{b}$, and $z_{b}$ are always positive while $z_{c}$ and $z$ are always negative. The angle $q$ is negative with respect to the $y$ direction shown in Figure 2.

Figure 3 shows the workpiece coordinate system $\left(x_{w} \cdot y_{w}, z_{w}\right)$ and the reference coordinate system (XYZ). Using geometric transformation to transform the workpiece coordinate to the reference coordinate as follows:

$$
\left\{\mathrm{O}_{\mathrm{w}}\right\} \xrightarrow{\mathrm{T}}\left\{\mathrm{O}_{1}\right\} \xrightarrow{\mathrm{T}}\left\{\mathrm{O}_{2}\right\} \xrightarrow{\mathrm{R}}\left\{\mathrm{O}_{2}\right\} \xrightarrow{\mathrm{T}}\{\mathrm{O}\}
$$

where $T$ means translation and $R$ means rotation.
We will get:

$$
\begin{aligned}
& X=x_{w}+x_{O w}^{O 1}+p \\
& Y=\left(y_{w}+y_{O w}^{O 1}\right) \times \cos a-\left(z_{w}+z_{O 1}^{O 2}\right) \times \sin a
\end{aligned}
$$

(3b)

$$
\begin{equation*}
Z=\left(y_{w}+y_{O w}^{O 1}\right) \times \sin a+\left(z_{w}+z_{O 1}^{O 2}\right) \times \cos a-h \tag{3c}
\end{equation*}
$$

Where $x_{O_{w}}^{O_{1}} y_{O_{w}}^{O_{1}} z_{O_{w}}^{O_{1}}$ are the coordinates with the origin at $O_{w}$ written in frame $O_{1}$, and $z_{O 1}^{O 2}$ is the coordinate of the origin $O_{1}$ in frame $O_{2}$.


Figure 3. The Workpiece Table Reference Position and Intermediate Reference Systems

During cutting, the end of the tooltip ( $\{\mathrm{P}\}$ ) has to be located on the surface of the workpiece. So from equations (1), (1a), (1b), (1c) and (3a) - (3b), finally, the solution for $l_{1}$ can be written as following:
$l_{1}=\left(x_{w}+x_{\text {Ow }}^{01}+p\right)+\left(T_{L}+2 c\right) \sin (q)+w+$
$\sqrt{\begin{array}{l}R^{2}-\left[-\left(y_{w}+y_{o w}^{o l}\right) \cos a+\left(z_{w}+z_{o 1}^{o 2}\right) \sin a-b+a-e\right]^{2}- \\ {\left[-\left(y_{w}+y_{o w}^{o 1}\right) \sin a-\left(z_{w}+z_{o 1}^{o 2}\right) \cos a+h-\left(T_{L}+2 c\right) \cos (q)+d-g e\right]^{2}}\end{array}}$
Similarly, we can solve $l_{2}, l_{3}$, and $l_{4}$ as follows:
$l_{2}=\left(x_{w}+x_{\mathrm{ov}}^{\mathrm{O}}+p\right)+\left(T_{L}+2 c\right) \sin (q)-w-$
$\sqrt{\begin{array}{l}R^{2}-\left[-\left(y_{w}+y_{o w}^{o l}\right) \cos a+\left(z_{w}+z_{o 1}^{o 2}\right) \sin a-b+a-e\right]^{2}- \\ {\left[-\left(y_{w}+y_{o w}^{o 1}\right) \sin a-\left(z_{w}+z_{o 1}^{o 2}\right) \cos a+h-\left(T_{L}+2 c\right) \cos (q)+d-g e\right]^{2}}\end{array}}$
$l_{3}=\left(x_{w}+x_{\text {ow }}^{01}+p\right)+T_{L} \sin (q)+w+$
$\sqrt{\begin{array}{l}R^{2}-\left[\left(y_{w}+y_{O w}^{O 1}\right) \cos a-\left(z_{w}+z_{o 1}^{O 2}\right) \sin a-b+a-e\right]^{2}- \\ {\left[-\left(y_{w}+y_{o w}^{o 1}\right) \sin a-\left(z_{w}+z_{o 1}^{O 2}\right) \cos a+h-T_{L} \cos (q)-d-g e\right]^{2}}\end{array}}$
$I_{4}=\left(x_{w}+x_{o w}^{O 1}+p\right)+T_{L} \sin (q)-w-$
$\sqrt{\begin{array}{l}R^{2}-\left[\left(y_{w}+y_{O w}^{O 1}\right) \cos a-\left(z_{w}+z_{O 1}^{O 2}\right) \sin a-b+a-e\right]^{2}- \\ {\left[-\left(y_{w}+y_{o w}^{O 1}\right) \sin a-\left(z_{w}+z_{o 1}^{O 2}\right) \cos a+h-T_{L} \cos (q)-d-g e\right]^{2}}\end{array}}$

For the tooltip orientation, it has to be coincident with the direction-cosine of cutting position on the workpiece surface.


Figure 4. ToolTip orientation and workpiece direction cosine

The unit vector of the normal plane at cutting location of a workpiece surface can be written as $I_{w} i_{w}+J_{w} i_{w}+K_{w} k_{w}$, where $I_{w}, J_{w}, K_{w}$ are the direction cosine. So, the orientation can be obtained as follows:

$$
\begin{aligned}
& p=\arctan \left(\frac{K_{w}}{J_{w}}\right) ; I_{2}=I_{w} \\
& J_{2}=\left(\sqrt{J_{w}{ }^{2}+K_{w}{ }^{2}}\right) \cos (a+p) \\
& K_{2}=\left(\sqrt{J_{w}{ }^{2}+K_{w}{ }^{2}}\right) \sin (a+p)
\end{aligned}
$$

From these equations, the solution of $a$ and $q$ will be:

$$
\begin{aligned}
\alpha & = \pm 90^{\circ}-\arctan \left(\frac{K_{w}}{J_{w}}\right) \text { where }-90^{\circ} £ \operatorname{a} £ 90^{\circ} \\
\theta & =\arctan \left(\frac{I_{w}}{\sqrt{J_{w}^{2}+K_{w}^{2}}}\right) \quad \text { where- } 90^{\circ} £ q £ 90^{\circ}
\end{aligned}
$$

## Forward or Direct Kinematics

In the previous section, we derive the robot variables, $l_{1}, l_{2}, l_{3}, l_{4}$ and $a$ from a given position and orientation on a workpiece surface. In this section we will derive the tooltip position and orientation from a given set of robot variables. The robot variables $l_{1}, l_{2}, l_{3}, l_{4}$ and $a$, shown in Figure 3a and 3b, will be used to define new variables depends as,

$$
r_{1}=\frac{l_{1}+l_{2}}{2}, r_{2}=\frac{l_{3}+l_{4}}{2}, d_{1}=l_{1}-l_{2}, d_{2}=l_{3}-l_{4}
$$

So, the rotation angle of the platform can readily be determined as

$$
\begin{equation*}
\cos q=\frac{\sqrt{4 c^{2}-\left(r_{2}-r_{1}\right)^{2}}}{2 c} \tag{4}
\end{equation*}
$$

Position relationship can be obtained by manipulating expression (1), (2), (3) and (4) which lead to,
$\left(l_{1}-x_{a}-w\right)^{2}+\left(l_{3}-x_{b}-w\right)^{2}=2 R^{2}-2 y_{c}^{2}-2(b-a+e)^{2}$
$-2\left(z_{c}+g e\right)^{2}-2(c \times \cos q-d)^{2}$
$\left(l_{2}-x_{a}-w\right)^{2}-\left(l_{4}-x_{b}-w\right)^{2}=-4 y_{c}(b-a-e)$
$-4\left(z_{c}+g e\right)(c \times \cos q-d)$
(4b)


Figure 5. New variables defined


Figure 6. New variables defined
Simplify the expressions above, using the new variables shown in Figure 5-6, we will have

$$
\begin{aligned}
& \left(l_{2}-x_{a}\right)^{2}=\left(l_{1}-x_{a}\right)^{2}=\frac{d_{1}^{2}}{4} \\
& \left(l_{4}-x_{b}\right)^{2}=\left(l_{3}-x_{b}\right)^{2}=\frac{d_{2}^{2}}{4}
\end{aligned}
$$

From equation (1a) and Figure 5-6, the movement in $x$ direction is easily obtain as

$$
\begin{equation*}
x=\left(\frac{r_{1}+r_{2}}{2}\right)-\left(c+T_{L}\right) \sin \theta \tag{5}
\end{equation*}
$$

which finally, we will obtain

$$
\begin{equation*}
y=y_{c}=\frac{d_{1}^{2}-d_{2}^{2}-4 w\left(d_{1}-d_{2}\right)-16(d-c \cdot \cos q)\left(z_{c}+g e\right)}{16(a-b-e)} \tag{6}
\end{equation*}
$$

We can also obtain $Z_{c}$ by substitutes the above expression in (4b), which leading to $2^{\text {nd }}$ degree polynomial equation as:

$$
\begin{align*}
& \frac{16^{2}(d-c \cos \theta)^{2} \cdot g e^{2}-32 \cdot g e \cdot(d-c \cos \theta)\left(d_{1}^{2}-d_{2}^{2}-4 u\left(d_{1}-d_{2}\right)\right)}{128(a-b-c)^{2}} \\
& +\frac{\left(d_{1}^{2}-d_{2}^{2}-4 u\left(d_{1}-d_{2}\right)\right)^{2}}{128(a-b-e)^{2}}+2 u^{2}-u\left(d_{1}+d_{2}\right)+2 \cdot g e^{2}  \tag{7}\\
& +\frac{\left(d_{1}^{2}+d_{2}^{2}\right)}{4}-2 R^{2}+2(b-a)^{2}-4 e(a-b)+2 e^{2}+2(c \cdot \cos \theta-d)^{2} \\
& +z_{0}\left(4 \cdot g e-\frac{32(d-c \cos \theta)\left(d_{1}^{2}-d_{2}^{2}-4 u\left(d_{1}-d_{2}\right)\right)-512 \cdot g e \cdot(d-c \cos \theta)^{2}}{128(a-b-e)^{2}}\right) \\
& +z_{0}^{2}\left(\frac{16^{2}(d-c \cos \theta)^{2}}{128(a-b-e)^{2}}+2\right)=0
\end{align*}
$$

From equation (7), we can solve with simple formula:

$$
\begin{equation*}
z_{c}=\frac{-B \pm \sqrt{B^{2}-4 A C}}{2 A} \tag{8}
\end{equation*}
$$

Due to configuration of robot, the solution in $Z_{c}$ is always in negative. Therefore, the coefficient $A, B, C$ can be found as:

$$
A=\left(\frac{16^{2}(d-c \cos \theta)^{2}}{128(a-b-e)^{2}}+2\right)
$$

$B=4 . g e+\frac{512 . g e .(d-c \cos q)^{2}}{128(a-b-e)^{2}}$
$-\frac{32(d-c \cos q)\left(d_{1}^{2}-d_{2}^{2}-4 w\left(d_{1}-d_{2}\right)\right)}{128(a-b-e)^{2}}$
$C=\frac{16^{2}(d-c \cos q)^{2} \cdot g e^{2}}{128(a-b-e)^{2}}$
$-\frac{32 . g e .(d-c \cos q)\left(d_{1}^{2}-d_{2}^{2}-4 w\left(d_{1}-d_{2}\right)\right)}{128(a-b-e)^{2}}$
$+\frac{\left(d_{1}^{2}-d_{2}^{2}-4 w\left(d_{1}-d_{2}\right)\right)^{2}}{128(a-b-e)^{2}}+2 w^{2}-w\left(d_{1}+d_{2}\right)+2 . g e^{2}+2 e^{2}$
$+\frac{\left(d_{1}^{2}+d_{2}^{2}\right)}{4}-2 R^{2}+2(b-a)^{2}-4 e(a-b)+2(c \cdot \cos q-d)^{2}$
Knowing $Z_{c}$ from equation (8), $Z$ can be found as

$$
z=z_{c}-\left(c+T_{L}\right) \cos q
$$

Using geometric transformation to transform the reference coordinate to the workpiece coordinate as follows:

$$
\{\mathrm{O}\} \xrightarrow{\mathrm{T}}\left\{\mathrm{O}_{2}\right\} \xrightarrow{\mathrm{R}}\left\{\mathrm{O}_{2}\right\} \xrightarrow{\mathrm{T}}\left\{\mathrm{O}_{1}\right\} \xrightarrow{\mathrm{T}}\left\{\mathrm{O}_{\mathrm{w}}\right\}
$$

So, tool-tip contact point on the workpiece surface can be obtained by equations as follow:

$$
\begin{aligned}
& x_{w}=x+p+x_{O 1}^{O w} \\
& y_{w}=y \times \cos (a)+(z+h) \times \sin (a)+y_{O 1}^{O w} \\
& z_{w}=(z+h) \times \cos (a)-y \times \sin (a)+z_{O 2}^{O 1}
\end{aligned}
$$

Finally, we can find direction cosine of the workpiece cutting location, or tool-tip contact point, from angle $\theta$ and $\alpha$ as:

$$
\begin{aligned}
& I_{w}= \pm \frac{1}{\sqrt{1+\frac{1}{\tan ^{2} q}}} \\
& \text { where } \operatorname{sign}\left(I_{w}\right)=\operatorname{sign}(q) \\
& J_{w}= \pm \frac{1}{\left(\sqrt{1+\tan ^{2} q}\right) \times\left(\sqrt{1+\tan ^{2}(90-a)}\right)}
\end{aligned}
$$

where $\operatorname{sign}\left(J_{w}\right)=\operatorname{sign}(a)$
$K_{w}=\frac{\tan (90-a)}{\left(\sqrt{1+\tan ^{2} q}\right) \times\left(\sqrt{1+\tan ^{2}(90-a)}\right)}$
where $\operatorname{sign}\left(K_{w}\right)=+$
In order to obtain the velocity relationship between the joint variable and the working coordinate, we apply the differential position relationship instead. Or knowing the velocity vector on the platform relative to the rotating table directly, we can obtain the relationship of the velocity of the tool attached relative to the moving table which represents by $\mathrm{x}_{\mathrm{T}}^{\mathrm{TP}}=\left[\begin{array}{lllll}x_{w} & y_{w} & z_{w} & \theta & \alpha\end{array}\right]^{T}$ and the actuator input denoted by $\mathbf{q}$ as following:

Moving platform velocity,

$$
\mathbf{V}_{\mathrm{Bi}}=\mathbf{V}_{\mathbf{P}}+\dot{\theta} j \times \mathbf{P B}_{\mathbf{i}}
$$

Then, the velocity at point B related to that of point A,

$$
\mathbf{A}_{\mathbf{i}} \mathbf{B}_{\mathbf{i}} \cdot \mathbf{V}_{\mathrm{Ai}}=\mathbf{A}_{\mathbf{i}} \mathbf{B}_{\mathbf{i}} \cdot \mathbf{V}_{\mathrm{Bi}}
$$

According to [5], the Jacobian matrix of the machine can be written in the form as

$$
\mathbf{A} \dot{\mathbf{q}}=\mathbf{B} \dot{\mathbf{x}} \quad \text { or } \quad \dot{\mathbf{q}}=J \dot{\mathbf{x}} \text { where } \mathbf{J}=\mathbf{A}^{-1} \mathbf{B}
$$

So,

$$
\begin{gathered}
\mathbf{A}_{\mathbf{i}} \mathbf{B}_{\mathbf{i}} \bullet V_{A i}=A_{i} B_{i} \bullet V_{P / w}-A_{i} B_{i} \bullet \alpha\left(\sqrt{y_{w}^{2}+\left(z_{w}+Z_{01}^{02}\right)^{2}}\right) i \\
+\left(P B_{i} \times A_{i} B_{i}\right) \bullet \dot{\theta} j
\end{gathered}
$$

$$
\mathbf{A}=\left[\begin{array}{ccccc}
\mathbf{A}_{1} \mathbf{B}_{1} \bullet \mathrm{i} & 0 & 0 & 0 & 0 \\
0 & \mathbf{A}_{2} \mathbf{B}_{2} \bullet \mathrm{i} & 0 & 0 & 0 \\
0 & 0 & \mathbf{A}_{3} \mathbf{B}_{3} \bullet \mathbf{i} & 0 & 0 \\
0 & 0 & 0 & \mathbf{A}_{4} \mathbf{B}_{4} \bullet \mathbf{i} & 0 \\
0 & 0 & 0 & 0 & \mathbf{i}
\end{array}\right]
$$

$$
\dot{\mathbf{q}}=\left[\begin{array}{lllll}
\dot{l}_{1} & \dot{l}_{2} & \dot{l}_{3} & \dot{l}_{4} & \dot{\alpha}
\end{array}\right]^{T}
$$

$$
\dot{\mathbf{x}}_{T}^{T P}=\left[\begin{array}{lllll}
\dot{x_{w}} & \dot{y_{w}} & \dot{z}_{w} & \dot{\theta} & \dot{\alpha} \tag{9}
\end{array}\right]^{T}
$$

To derive the singularity, we consider only H-4 parallel manipulator which matrix $A$ and $B$ reduce to $4 x 4$ matrix as follows[9]:

$$
\begin{aligned}
& \mathbf{A}=\left[\begin{array}{cccc}
\mathbf{A}_{1} \mathbf{B}_{1} \bullet \mathbf{i} & 0 & 0 & 0 \\
0 & \mathbf{A}_{2} \mathbf{B}_{2} \bullet \mathbf{i} & 0 & 0 \\
0 & 0 & \mathbf{A}_{3} \mathbf{B}_{3} \bullet \mathbf{i} & 0 \\
0 & 0 & 0 & \mathbf{A}_{4} \mathbf{B}_{4} \bullet \mathbf{i}
\end{array}\right] \\
& \mathbf{B}=\left[\begin{array}{ll}
\mathbf{A}_{1} \mathbf{B}_{1} \bullet u_{p} & \left(\mathbf{A}_{1} \mathbf{B}_{1} \times \mathbf{P} \mathbf{B}_{1}\right) \bullet \mathbf{j} \\
\mathbf{A}_{2} \mathbf{B}_{2} \bullet u_{p} & \left(\mathbf{A}_{2} \mathbf{B}_{2} \times \mathbf{P} \mathbf{B}_{2}\right) \bullet \mathbf{j} \\
\mathbf{A}_{3} \mathbf{B}_{3} \bullet u_{p} & \left(\mathbf{A}_{3} \mathbf{B}_{3} \times \mathbf{P B}_{3}\right) \bullet \mathbf{j} \\
\mathbf{A}_{4} \mathbf{B}_{4} \bullet u_{p} & \left(\mathbf{A}_{4} \mathbf{B}_{4} \times \mathbf{P B}_{4}\right) \bullet \mathbf{j}
\end{array}\right]
\end{aligned}
$$

At this point one can derive the singularity of the manipulator, according to [5] and [1], by finding the determinant of matrix $\mathbf{A}$ and $\mathbf{B}$. In this case we interests in singularity of type 2 [5] which occurs within workspace. From the set of equations (9), substitute all parameters into the matrix yield,

$$
\mathbf{B}=\left[\begin{array}{lllll}
x_{a}-l_{1} & y_{a} & z_{a} & {\left[\begin{array}{ll}
x_{a}-l_{1} & z_{a}
\end{array}\right] \mathbf{u}_{1}}  \tag{10}\\
x_{a}-l_{2} & y_{a} & z_{a} & {\left[\begin{array}{ll}
x_{a}-l_{2} & z_{a}
\end{array}\right] \mathbf{u}_{1}} \\
x_{b}-l_{3} & y_{b} & z_{b} & {\left[\begin{array}{ll}
x_{b}-l_{3} & z_{b}
\end{array}\right] \mathbf{u}_{2}} \\
x_{b}-l_{4} & y_{b} & z_{b} & {\left[\begin{array}{ll}
x_{b}-l_{4} & z_{b}
\end{array}\right] \mathbf{u}_{2}}
\end{array}\right]
$$

where,

$$
\begin{aligned}
& \mathbf{u}_{1}=\left[\begin{array}{ll}
-c \cdot \cos \theta & c \cdot \sin \theta
\end{array}\right]^{T} \\
& \mathbf{u}_{2}=\left[\begin{array}{ll}
c \cdot \cos \theta & -c \cdot \sin \theta
\end{array}\right]^{T}
\end{aligned}
$$

The analytical form of the determinant of matrix $\mathbf{B}$ is found to be,

$$
\operatorname{det}(\mathbf{B})=2 \cdot c \cdot\left(l_{1}-l_{2}\right) \cdot\left(l_{3}-l_{4}\right) \cdot \cos \theta \cdot[2 y(d-\cos \theta)-2 z(b-a)]
$$

(11)

From observation of equation (11), the singularity occurs at following cases,

1) $l_{1}-l_{2}$ or $l_{3}-l_{4}$ is zero. This indicate that lines $\mathbf{A}_{1} \mathbf{B}_{1}$ and $\mathbf{A}_{2} \mathbf{B}_{2}$ or $\mathbf{A}_{3} \mathbf{B}_{3}$ and $\mathbf{A}_{4} \mathbf{B}_{4}$ are coincide.
2) $\cos q=0$ or $q= \pm 90^{\circ}$. This means that the moving platform is parallel to x -axis.
3) $y(d-\cos q)-z(b-a)=0 \mathrm{P} \quad z=\frac{y(d-\cos q)}{(b-a)}$,
which shows that the planes formed by lines $\mathbf{A}_{1} \mathbf{B}_{1}$ and $\mathbf{A}_{2} \mathbf{B}_{2}$ and by $\mathbf{A}_{3} \mathbf{B}_{3}$ and $\mathbf{A}_{4} \mathbf{B}_{4}$ are in parallel.

(1), (2)

(3)

Figure 7. Singularity Configurations
From equation $\dot{\mathbf{q}}=\mathbf{J} \dot{\mathbf{x}}$, we can find Jacobian matrix $(\mathbf{J})$ by differentiate inverse kinematics equation as follow:

$$
\begin{aligned}
& {\left[\begin{array}{lll}
\frac{\partial f_{1}}{\partial x_{w}} & \frac{\partial f_{1}}{\partial y_{w}} & \frac{\partial f_{1}}{\partial z_{w}}
\end{array} \frac{\partial f_{1}}{\partial \theta} \frac{\partial f_{1}}{\partial \alpha}\right]} \\
& \left.\frac{\partial f_{2}}{\partial x} \frac{\partial f_{2}}{\partial y_{w}} \frac{\partial f_{2}}{\partial z_{w}} \frac{\partial f_{2}}{\partial \theta} \frac{\partial f_{2}}{\partial \alpha} \right\rvert\, \text {, where } \\
& J=\frac{\partial f_{3}}{\partial x_{w}} \frac{\partial f_{3}}{\partial y_{w}} \frac{\partial f_{3}}{\partial z_{w}} \frac{\partial f_{3}}{\partial \theta} \frac{\partial f_{3}}{\partial \alpha} \\
& \frac{\partial f_{4}}{\partial x_{w}} \frac{\partial f_{4}}{\partial y_{w}} \frac{\partial f_{4}}{\partial z_{w}} \frac{\partial f_{4}}{\partial \theta} \frac{\partial f_{4}}{\partial \alpha} \\
& \frac{\partial f_{5}}{\partial x_{w}} \frac{\partial f_{5}}{\partial y_{w}} \frac{\partial f_{5}}{\partial z_{w}} \frac{\partial f_{5}}{\partial \theta} \frac{\partial f_{5}}{\partial \alpha} \\
& \frac{\partial f_{1}}{\partial x_{w}}=\frac{\partial f_{2}}{\partial x_{w}}=\frac{\partial f_{3}}{\partial x_{w}}=\frac{\partial f_{4}}{\partial x_{w}}=\frac{\partial f 5}{\partial \alpha}=1 \text { and } \\
& \frac{\partial f_{s}}{\partial x_{w}}=\frac{\partial f_{s}}{\partial y_{w}}=\frac{\partial f_{s}}{\partial z_{w}}=\frac{\partial f_{s}}{\partial \theta}=0 \\
& \frac{\partial f_{1}}{\partial y_{\varepsilon}}=-A^{-\frac{1}{2}}(B \cos \alpha+C \sin \alpha) \\
& \frac{\partial f_{1}}{\partial z_{\text {e }}}=A^{-\frac{1}{2}}(B \cos \alpha-C \sin \alpha) \\
& \frac{\partial f_{1}}{\partial \theta}=\left(T_{L}+2 c\right)\left(\cos \theta+A^{-\frac{1}{2}} C \sin \theta\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial f_{1}}{\partial \alpha}=A^{-\frac{1}{2}}\left[\begin{array}{l}
B\left[\left(y_{\varepsilon}+Y_{0 w}^{01}\right) \sin \alpha+\left(z_{\varepsilon}+Z_{01}^{02}\right) \cos \alpha\right]+ \\
C\left[\left(z_{v}+Z_{01}^{02}\right) \sin \alpha-\left(y_{\varepsilon}+Y_{0 w}^{01}\right) \cos \alpha\right]
\end{array}\right) \\
& \frac{\partial f_{2}}{\partial y_{w}}=A^{-\frac{1}{2}}(B \cos \alpha+C \sin \alpha) \\
& \frac{\partial f_{2}}{\partial z_{v}}=-A^{-\frac{1}{2}}(B \sin \alpha-C \cos \alpha) \\
& \frac{\partial f_{s}}{\partial \theta}=\left(T_{L}+2 c\right)\left(\cos \theta-A^{-\frac{1}{2}} C \sin \theta\right) \\
& \frac{\partial f_{z}}{\partial \alpha}=-A^{-\frac{1}{2}}\binom{B\left[\left(y_{\varepsilon}+Y_{0 w}^{01}\right) \sin \alpha+\left(z_{\varepsilon}+Z_{01}^{02}\right) \cos \alpha\right]+}{C\left[\left(z_{\varepsilon}+Z_{01}^{02}\right) \sin \alpha-\left(y_{\varepsilon}+Y_{0 w}^{01}\right) \cos \alpha\right]}
\end{aligned}
$$

Where $\quad A=R^{2}-B^{2}-C^{2}$

$$
\begin{aligned}
& B=\left(y_{w}+y_{O w}^{O 1}\right) \cos \alpha-\left(z_{w}+z_{O 1}^{O 2}\right) \sin \alpha+b-a+e \\
& C=\left(y_{w}+y_{O w}^{O 1}\right) \sin \alpha+\left(z_{w}+z_{O 1}^{O 2}\right) \cos \alpha \\
& -h+\left(T_{L}+2 c\right) \cos (\theta)-d+g e
\end{aligned}
$$

$$
\frac{\partial f_{3}}{\partial y_{w}}=-A_{1}^{-\frac{1}{2}}\left(B_{1} \cos \alpha+C_{1} \sin \alpha\right)
$$

$$
\frac{\partial f_{3}}{\partial z_{w}}=A_{1}^{-\frac{1}{2}}\left(B_{1} \sin \alpha-C_{1} \cos \alpha\right)
$$

$$
\frac{\partial f_{s}}{\partial \theta}=T_{L}\left(\cos \theta+A_{1}^{-\frac{1}{2}} C_{1} \sin \theta\right)
$$

$$
\begin{aligned}
& \frac{\partial f_{s}}{\partial \alpha}=A_{1}^{-\frac{1}{2}}\binom{B_{1}\left[\left(y_{w}+Y_{0 w}^{01}\right) \sin \alpha+\left(z_{w}+Z_{01}^{02}\right) \cos \alpha\right]+}{C_{1}\left[\left(z_{w}+Z_{01}^{02}\right) \sin \alpha-\left(y_{w}+Y_{0 w}^{01}\right) \cos \alpha\right]} \\
& \frac{\partial f_{4}}{\partial y_{w}}=A_{1}^{-\frac{1}{2}}\left(B_{1} \cos \alpha+C_{1} \sin \alpha\right) \\
& \frac{\partial f_{4}}{\partial z_{w}}=-A_{1}^{-\frac{1}{2}}\left(B_{1} \sin \alpha-C_{1} \cos \alpha\right) \\
& \frac{\partial f_{4}}{\partial \theta}=T_{L}\left(\cos \theta-A_{1}^{-\frac{1}{2}} C_{1} \sin \theta\right)
\end{aligned}
$$

$\frac{\partial f_{4}}{\partial \alpha}=-A_{1}^{-\frac{1}{2}}\binom{B_{1}\left[\left(y_{\omega}+Y_{0 w}^{01}\right) \sin \alpha+\left(z_{\omega}+Z_{01}^{02}\right) \cos \alpha\right]+}{C_{1}\left[\left(z_{\omega}+Z_{01}^{02}\right) \sin \alpha-\left(y_{\omega}+Y_{0 w}^{01}\right) \cos \alpha\right]}$
Where $\quad A_{1}=R^{2}-B_{1}{ }^{2}-C_{1}{ }^{2}$
$B_{1}=\left(y_{w}+y_{O w}^{O 1}\right) \cos a-\left(z_{w}+z_{O 1}^{O 2}\right) \sin a-b+a-e$
$C_{1}=\left(y_{w}+y_{O w}^{O 1}\right) \sin a+\left(z_{w}+z_{O 1}^{O 2}\right) \cos a$
$-h+T_{L} \cos (q)+d+g e$

## 4. Analytical Dynamics Model

In this section, the conservative Lagrange's equation is used to obtain the dynamic model of the closed-chain H-4 type from the forward kinematics derived earlier. For conservative system, the well-known generalized Lagrange's equation of coordinates $q_{i}$, with generalized force $Q_{i}$, can be written as:

$$
\frac{d}{d t} \frac{\partial(K \cdot E .)}{\partial \dot{q}_{i}}-\frac{\partial(K \cdot E .)}{\partial q_{i}}+\frac{\partial(P \cdot E .)}{\partial q_{i}}+\frac{\partial(D \cdot E .)}{\partial q_{i}}=Q_{i}
$$

The robot manipulator consists of 5 components as shown in Figure 8 which can be detailed as following,
-Milling Head or Platform 1 part.
-Connecting Head
-Arm or link
-Linear joint
-Universal cross (U-joint)
We can write the Lagrange's equation of each component above. Considering the platform component, the Lagrange's equation based on the coordinate of the linear joint $\boldsymbol{I}_{\boldsymbol{i}}$ of 4-DOF can be derived as following:

$$
\begin{align*}
\frac{d}{d t} \frac{\partial\left(K \cdot E_{p}\right)}{\partial i_{i}}-\frac{\partial\left(K \cdot E_{p}\right)}{\partial l_{i}}+\frac{\partial\left(P \cdot E_{p}\right)}{\partial l_{i}} & =F_{i p}  \tag{12}\\
& i=1,2,3,4
\end{align*}
$$

where

For the two connecting heads, the Lagrange's equation will be:

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial\left(K \cdot E_{O H}\right)}{\partial \dot{i}_{i}}-\frac{\partial\left(K \cdot E_{O H}\right)}{\partial l_{i}}+\frac{\partial\left(P \cdot E_{O H}\right)}{\partial l_{i}}=F_{i O H} \tag{13}
\end{equation*}
$$

where

$$
\frac{d}{d t} \frac{\partial\left(K E_{\text {CHH }}\right)}{\partial \dot{l}_{i}}=m_{\text {CH }}\binom{\dot{x}_{\text {CHom }} \frac{d}{d t} \frac{\partial \dot{x}_{\text {CHem }}}{\partial \dot{l}_{i}}+\ddot{x}_{\text {CHcm }} \frac{\partial \dot{x}_{\text {CHom }}}{\partial \dot{l}_{i}}+\dot{y}_{\text {CHom }} \frac{d}{d t} \frac{\partial \dot{y}_{\text {CHom }}}{\partial \dot{l}_{i}}}{+\ddot{y}_{\text {CHom }} \frac{\partial \ddot{y}_{\text {CHcm }}}{\partial \dot{l}_{i}}+\dot{z}_{\text {CHom }} \frac{d}{d t} \frac{\partial \dot{z}_{\text {CHam }}}{\partial \dot{l}_{i}}+\ddot{z}_{\text {CHcm }} \frac{\partial \ddot{z}_{\text {CHcm }}}{\partial \dot{l}_{i}}}
$$

$$
\frac{\partial\left(K E_{C H}\right)}{\partial l_{i}}=m_{\text {CH }}\left(\dot{x}_{\text {CHm }} \frac{\partial \dot{x}_{\text {CHm }}}{\partial l_{i}}+\dot{y}_{\text {CHm }} \frac{\partial \dot{y}_{\text {CHmm }}}{\partial l_{i}}+\dot{\tilde{z}}_{\text {CHem }} \frac{\partial \dot{z}_{\text {CHmm }}}{\partial l_{i}}\right)
$$

$$
\begin{equation*}
\frac{\partial\left(P \cdot E_{\text {OH }}\right)}{\partial l_{i}}=m_{\text {OH }} 9810 \frac{\partial z_{\text {OHom }}}{\partial l_{i}} \tag{13a}
\end{equation*}
$$

$$
\begin{align*}
& +\bar{I}_{\text {MAHy }}\left(\dot{\theta} \frac{d}{d t} \frac{\partial \dot{\theta}}{\partial \dot{l}_{i}}+\ddot{\theta} \frac{\partial \dot{\theta}}{\partial \dot{l}_{i}}\right) \\
& \frac{\partial\left(K \cdot E_{p}\right)}{\partial l_{i}}=m_{M A}\left(\dot{x}_{M H o m} \frac{\partial \dot{x}_{\text {MHem }}}{\partial l_{i}}+\dot{y}_{M H o m} \frac{\partial \dot{y}_{\text {MHem }}}{\partial l_{i}}+\dot{z}_{\text {MHom }} \frac{\partial \dot{z}_{\text {MHem }}}{\partial l_{i}}\right)+\bar{I}_{M H H_{i}} \dot{\theta} \frac{\partial \dot{\theta}}{\partial l_{i}} \\
& \frac{\partial\left(P \cdot E_{p}\right)}{\partial l_{i}}=m_{M H} 9810 \frac{\partial z_{\text {MHFom }}}{\partial l_{i}} \tag{12a}
\end{align*}
$$

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial\left(K \cdot E_{A m}\right)}{\partial \dot{l}_{i}}-\frac{\partial\left(K \cdot E_{A m}\right)}{\partial l_{i}}+\frac{\partial\left(P \cdot E_{A m}\right)}{\partial l_{i}}=F_{i A m} \tag{14}
\end{equation*}
$$

where

$$
\begin{align*}
& \frac{d}{d t} \frac{\partial\left(K \cdot E_{A_{A m}}\right)}{\partial \dot{i}_{i}}=m_{A m}\binom{\dot{x}_{\text {Amom }} \frac{d}{d t} \frac{\partial \dot{x}_{\text {Amam }}}{\partial \dot{i}_{i}}+\ddot{x}_{\text {Amom }} \frac{\partial \dot{x}_{\text {Amom }}}{\partial \dot{i}_{i}}+\dot{y}_{\text {Amom }} \frac{d}{d t} \frac{\partial \dot{y}_{\text {Amem }}}{\partial \dot{i}_{i}}}{+\ddot{y}_{\text {Amom }} \frac{\partial \ddot{y}_{\text {Amem }}}{\partial \dot{i}_{i}}+\dot{z}_{\text {Amom }} \frac{d}{d t} \frac{\partial \dot{z}_{\text {Amom }}}{\partial \dot{i}_{i}}+\ddot{z}_{\text {Amem }} \frac{\partial \ddot{z}_{\text {Amcm }}}{\partial \dot{i}_{i}}} \\
& +\bar{I}_{X X, A m}\left(\dot{\varphi}_{x} \frac{d}{d t} \frac{\partial \dot{\varphi}_{s}}{\partial \dot{l}_{i}}+\ddot{\varphi}_{s} \frac{\partial \dot{\varphi}_{x}}{\partial \dot{i}_{i}}\right)+\bar{I}_{Y Y, A m}\left(\dot{\varphi}_{y} \frac{d}{d t} \frac{\partial \dot{\varphi}_{y}}{\partial \dot{l}_{i}}+\ddot{\varphi}_{y} \frac{\partial \dot{\varphi}_{y}}{\partial \dot{l}_{i}}\right) \\
& \frac{\partial\left(K \cdot E_{A m}\right)}{\partial l_{i}}=m_{A m}\left(\dot{x}_{A m o m} \frac{\partial \dot{x}_{A m e m}}{\partial l_{i}}+\dot{y}_{A m o m} \frac{\partial \dot{y}_{\text {Amem }}}{\partial l_{i}}+\dot{z}_{A m e m} \frac{\partial \dot{z}_{\text {Amem }}}{\partial l_{i}}\right)+ \\
& +\bar{I}_{X X, A m}\left(\dot{\varphi}_{s} \frac{\partial \dot{\varphi}_{z}}{\partial l_{i}}\right)+\bar{I}_{Y Y, A m}\left(\dot{\varphi}_{y} \frac{\partial \dot{\varphi}_{y}}{\partial l_{i}}\right) \\
& \frac{\partial\left(P \cdot E_{A m}\right)}{\partial l_{i}}=m_{A m} 9810 \frac{\partial z_{A m m}}{\partial l_{i}} \tag{14a}
\end{align*}
$$

For the four linear joints, the Lagrange's equation will be:

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial\left(K \cdot E_{L J}\right)}{\partial i_{i}}=F_{i L J} \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial\left(K \cdot E_{L J}\right)}{\partial i_{i}}=m_{L J} \ddot{l}_{i} \tag{15a}
\end{equation*}
$$

For all universal crosses, the Lagrange's equation will be:

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial\left(K \cdot E_{U C}\right)}{\partial \dot{i}_{i}}-\frac{\partial\left(K \cdot E_{U C}\right)}{\partial l_{i}}+\frac{\partial\left(P \cdot E_{U C}\right)}{\partial l_{i}}=F_{i U C} \tag{16}
\end{equation*}
$$

where

$$
\begin{aligned}
& +\bar{I}_{u o q}\left(\dot{\beta} \frac{d}{d t} \frac{\partial \dot{\beta}}{\partial \dot{i}_{i}}+\ddot{\beta} \frac{\partial \dot{\beta}}{\partial \dot{i}_{i}}\right)
\end{aligned}
$$

$$
\begin{align*}
& \frac{\partial\left(P \cdot E_{U C}\right)}{\partial l_{i}}=m_{U O} 9810 \frac{\partial z_{U V_{0} m}}{\partial l_{i}} \tag{16a}
\end{align*}
$$

By Summing equations (12)-(16), we will get equation of motion of the system as:

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial(K \cdot E .)}{\partial i_{i}}-\frac{\partial(K \cdot E .)}{\partial l_{i}}+\frac{\partial(P \cdot E .)}{\partial l_{i}}=F_{i} \tag{16}
\end{equation*}
$$

According to the Figure 8, the parameters on connected head (CH) is found as: $g e=104 \mathrm{~mm}, e=$ 23.75 mm , $w=35 \mathrm{~mm}$


Figure 8. Robot Components
The parameters in equation (12a)-(12c) can be found as follows:

$$
\begin{aligned}
& x_{\text {MHEm }}=\left(\frac{1}{4}-\frac{E c}{4 c}\right) l_{1}+\left(\frac{1}{4}-\frac{E c}{4 c}\right) l_{2}+\left(\frac{1}{4}+\frac{E c}{4 c}\right) l_{3}+\left(\frac{1}{4}+\frac{E c}{4 c}\right) l_{4} \\
& \dot{x}_{\text {MFEm }}=\left(\frac{1}{4}-\frac{E c}{4 c}\right) \dot{l}_{1}+\left(\frac{1}{4}-\frac{E c}{4 c} \dot{\dot{l}}_{2}+\left(\frac{1}{4}+\frac{E c}{4 c}\right) \dot{d}_{3}+\left(\frac{1}{4}+\frac{E c}{4 c}\right) \dot{d}_{4}\right. \\
& \ddot{x}_{\text {MHFm }}=\left(\frac{1}{4}-\frac{E c}{4 c}\right) \ddot{l}_{1}+\left(\frac{1}{4}-\frac{E c}{4 c}\right) \ddot{l}_{2}+\left(\frac{1}{4}+\frac{E c}{4 c}\right) \ddot{l}_{3}+\left(\frac{1}{4}+\frac{E c}{4 c}\right) \ddot{l}_{4} \\
& \frac{\partial \dot{x}_{\text {MHcm }}}{\partial \dot{i}_{i}}=\left(\frac{1}{4}-\frac{E c}{4 c}\right) \quad \text { where } \quad i=1,2 ; \\
& \frac{\partial \dot{x}_{\text {MHEm }}}{\partial \dot{i}_{i}}=\left(\frac{1}{4}+\frac{E c}{4 c}\right) \quad \text { where } i=3,4 ;
\end{aligned}
$$

Due to the coefficient $A, B, C$ in equation (8), we can rearrange in terms of joint parameter $l_{1} l_{2} l_{3} l_{4}$ together with substitute the value of parameters $g e=104, e=$ $23.75, w=35$ and we can obtain


$$
\begin{aligned}
& z_{M H c m}=z_{0}-E c \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}} \\
& y_{M H c m}=\frac{\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)}{16(a-b-23.75)} \\
& -\frac{16\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}\right)\left(z_{c}+104\right)}{16(a-b-23.75)}
\end{aligned}
$$

The time derivative of $A, B, C$ can be found as

$$
\dot{\theta}=\left(\frac{\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}}{4 c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}\right)
$$

$$
\begin{aligned}
& \dot{A}=\left(\frac{d}{\left(4 c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right.}-0.25\right)\left(\frac{\left(l_{1}+l_{2}-l_{3}-l_{4}\right)\left(l_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)}{(a-b-23.75)^{2}}\right. \\
& \dot{B}=\frac{832\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{16 c(a-b-23.75)^{2} \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}
\end{aligned}
$$

$$
-\frac{\left(\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)\right)\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{64 c(a-b-23.75)^{2} \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}
$$

$$
\frac{\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)\left(\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-\dot{l}_{2}\right)-\left(l_{3}-l_{4}\right)\left(\dot{l}_{3}-\sqrt{l_{4}}\right)\right.}{\left.170\left(\dot{l}_{1}-\dot{l}_{2}-\dot{l}_{3}+\dot{l}_{4}\right)\right)} \text { 2(a-b-23.75)}
$$

$$
\dot{C}=\frac{2704\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{(a-b-23.75)^{2} c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}
$$

$$
\frac{26\left(\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)\right)\left(l_{1}+i_{2}-l_{3}-l_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{16(a-b-23.75)^{2} \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}
$$

$$
\frac{6656\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)\left(\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-\dot{l}_{2}\right)-\left(l_{3}-l_{4}\right)\left(\dot{l}_{3}-\dot{l}_{4}\right)-70\left(\dot{l}_{1}-\dot{l}_{2}-\dot{l}_{3}+\dot{l}_{4}\right)\right)}{128(a-b-23.75)^{2}}
$$

$$
+\frac{4\left(\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)\right)\left(\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-\dot{l}_{2}\right)-\left(l_{8}-l_{4}\right)\left(l_{3}-\dot{l}_{4}\right)-70\left(\dot{l}_{1}-\dot{l}_{2}-\dot{l}_{3}-\right.\right.}{128(a-b-23.75)^{2}}
$$

$$
-\left(c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}-d\right) \frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{4 c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}
$$

$$
+\frac{\left(\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-\dot{l}_{2}\right)+\left(l_{3}-l_{4}\right)\left(\dot{l}_{3}-\dot{l}_{4}\right)\right)}{2}-35\left(\dot{l}_{1}-\dot{l}_{2}+\dot{l}_{3}-\dot{l}_{4}\right)
$$

$$
\dot{z}_{c}=\frac{-\dot{B}}{2 A}-\frac{-B \dot{B}}{2 A \sqrt{B^{2}-4 A C}}+\frac{\dot{C}}{\sqrt{B^{2}-4 A C}}+\frac{\dot{A} C}{A \sqrt{B^{2}-4 A C}}+\frac{\dot{A} B}{2 A^{2}}+\frac{\dot{A}}{2 A^{2}} \sqrt{B^{2}-4 A C}
$$

$$
\dot{z}_{M H c m}=\dot{z}_{c}+E c \frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{16 c^{2} \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}}
$$

$$
\dot{y}_{M H c m}=\frac{\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-\dot{l}_{2}\right)-\left(l_{3}-l_{4}\right)\left(\dot{l}_{3}-\dot{l}_{4}\right)-70\left(\dot{l}_{1}-\dot{l}_{2}-\dot{l}_{3}+\dot{l}_{4}\right)}{8(a-b-23.75)}
$$

$$
\frac{-\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}\right) \dot{z}_{c}}{(a-b-23.75)}
$$

$$
-\frac{\left(z_{c}+104\right)\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{16 c(a-b-23.75) \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}}
$$

And the second time derivative of $A, B, C$ can be found as

$$
\begin{aligned}
& \ddot{\theta}=\left(\frac{\ddot{l}_{1}+\ddot{l}_{2}-\ddot{l}_{3}-\ddot{l}_{4}}{\sqrt{16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}}}\right)-\left(\frac{\left(l_{3}+l_{4}-l_{1}-l_{2}\right)\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}}{\left(16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}\right)^{\frac{3}{2}}}\right) \\
& \ddot{A}=\left(\frac{\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)}{(a-b-23.75)^{2}}\right) \frac{\left(l_{1}+l_{2}-l_{3}-l_{4}\right)\left(\ddot{l}_{1}+\ddot{l}_{2}-\ddot{l}_{3}-\ddot{l}_{4}\right)}{\sqrt{16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}}} \\
& +\left(\frac{\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)}{(a-b-23.75)^{2}}\right) \frac{\left(l_{1}+l_{2}-l_{3}-l_{4}\right)^{2}\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}}{\left(16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}\right)^{\frac{3}{2}}} \\
& +\left(\frac{\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)}{4 c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}\right) \frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}}{(a-b-23.75)^{2}} \\
& +\frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}\left(l_{1}+l_{2}-l_{3}-l_{4}\right)^{2}}{64 c^{2}\left(1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}\right)(a-b-23.75)^{2}} \\
& \ddot{B}=\left(\frac{208\left(d-c \sqrt{\left.1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}\right)}\right.}{(a-b-23.75)^{2}}\right) \frac{\left(l_{1}+l_{2}-l_{2}-l_{2}\right)\left(\vec{l}_{4}+\ddot{l}_{2}-\ddot{l}_{3}-\vec{l}_{4}\right)}{\sqrt{16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}}} \\
& +\left(\frac{208\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)}{(a-b-23.75)^{2}}\right) \frac{\left(l_{1}+l_{2}-l_{3}-l_{4}\right)^{2}\left(l_{1}+l_{2}-l_{3}-l_{4}\right)^{2}}{\left(16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}\right)^{\frac{3}{2}}} \\
& +\left(\frac{208\left(d-c \sqrt{1-\left(\frac{h_{1}+l_{2}}{4 c}-\frac{l_{2}+l_{4}}{4 c}\right)^{2}}\right)}{4 c \sqrt{1-\left(\frac{h_{2}+b_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}\right) \frac{\left(\dot{l}_{1}+\dot{i}_{2}-\dot{b}_{3}-i_{4}\right)^{2}}{(a-b-22.75)^{2}} \\
& +\frac{208\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}\left(l_{1}+l_{2}-l_{2}-l_{4}\right)^{2}}{64 c^{2}\left(1-\left(\frac{l_{2}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}\right)(a-b-23.75)^{2}} \\
& \frac{\left.\left(l_{1}+l_{2}-l_{3}-l_{4}\right)\left(\dot{l}_{1}+i_{2}-i_{3}-i_{4}\right)\left(\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-i_{2}\right)-\left(l_{2}-l_{4}\right)\left(i_{3}-i_{4}\right)-70 \dot{l}_{1}-i_{2}-\dot{l}_{3}+i_{4}\right)\right)}{32 c\left(\sqrt{1-\left(\frac{l_{4}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)(a-b-23.55)^{2}} \\
& \frac{\left(\left(l_{1}-l_{2}\right)^{2}-\left(l_{2}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)\right)}{16(a-b-23.75)^{2}\left(\sqrt{16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}}\right)}\left(\ddot{l}_{1}+\ddot{l}_{2}-\ddot{l}_{3}-\ddot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right) \\
& \frac{\left(\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)\right)}{16(a-b-23.75)^{2}\left(16 c^{2}-\left(l_{3}+l_{4}-l_{4}-l_{2}\right)^{2}\right)^{\frac{2}{2}}}\left(l_{1}+l_{2}-l_{3}-l_{4}\right)^{2}\left(l_{1}+\dot{l}_{2}-\dot{l}_{3}-l_{4}\right)^{2} \\
& \frac{\left(\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}-l_{3}+l_{4}\right)\right)}{64 c(a-b-23.75)^{2}\left(\sqrt{1-\left(\frac{l_{2}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)}\left(l_{1}+\dot{l}_{2}-l_{3}-l_{4}\right)^{2} \\
& \frac{\left(\left(l_{1}-l_{2}\right)\left(\ddot{l}_{1}-\ddot{l}_{2}\right)+\left(\dot{l}_{1}-\dot{l}_{2}\right)^{2}-\left(l_{3}-l_{4}\right)\left(\ddot{l}_{2}-\ddot{l}_{4}\right)-\left(\dot{l}_{3}-\dot{l}_{4}\right)^{2}-7\left(\ddot{l}_{1}-\ddot{l}_{2}-\ddot{l}_{3}+\ddot{l}_{4}\right)\right)}{2(a-b-23.75)^{2}\left(d-c \sqrt{1-\left(\frac{l_{2}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}\right)^{-1}} \\
& \left.\left.\frac{\left(\dot{l}_{1}+i_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{32 c(a-b-23.75)^{2} \sqrt{1-\left(\frac{l_{4}+l_{2}}{4 c}-\frac{l_{3}+l_{4}}{4 c}\right)^{2}}}\left(l_{1}-l_{2}\right)\left(i_{1}-i_{2}\right)-\left(l_{3}-l_{4}\right)\left(\dot{l}_{3}-i_{4}\right)-70 \dot{i}_{1}-\dot{i}_{2}-\dot{l}_{3}+i_{4}\right)\right)
\end{aligned}
$$

$\ddot{C}=\left(4 \mathrm{c}^{2} \dot{\theta} \sin ^{2}(\theta)-4 \mathrm{c}(\mathrm{c} \cos (\theta)-\mathrm{d}) \dot{\theta} \cos (\theta)\right) \dot{\theta}+\left(\frac{-6656 \mathrm{c} \sin (\theta)(I 2)-3328 \dot{\theta}(I 1) \mathrm{c} \cos (\theta)}{128(\mathrm{a}-\mathrm{b}-23.75)^{2}}\right) \dot{\theta}$

$+\left(-4 \mathrm{c}(\cos (\theta)-\mathrm{d}) \sin (\theta)+\frac{-3328(I 1) \cos (\theta)+5537792(\mathrm{~d}-\operatorname{cocos}(\theta)) \operatorname{csin}(\theta)}{128(\mathrm{a}-\mathrm{b}-23.75)^{2}}\right) \ddot{\theta}$
$+\left(\frac{\dot{d}_{1}}{2}+\frac{\left(4\left(2 d_{1}-140\right)(I 2)+4(I 1) \dot{d}_{1}-6656(d-c \cos (\theta)) \dot{d}_{1}-3328 \dot{\theta}\left(2 d_{1}-140\right) \operatorname{csin}(\theta)\right)}{128(\mathrm{a}-\mathrm{b}-23.75)^{2}}\right) \dot{d}_{1}$
$+\left(\frac{\dot{d}_{2}}{2}+\frac{\left(4\left(-2 d_{2}+140\right)(I 2)-4(I 1) \dot{d}_{2}+6656(d-c \cos (\theta)) \dot{d}_{2}-3328 \dot{\theta}\left(-2 d_{2}+140\right) \operatorname{csin}(\theta)\right)}{128(a-b-23.75)^{2}}\right) \dot{d}_{2}$
$+\left(\frac{d_{2}}{2}-35+\left(\frac{4(I 1)\left(-d_{2}+70\right)-6656(d-c \cos (\theta))\left(-d_{2}+70\right)}{128(a-b-23.75)^{2}}\right)\right) \ddot{d}_{2}$
$+\left(\frac{d_{1}}{2}-35+\left(\frac{4(I 1)\left(d_{1}-70\right)-6656(d-c \cos (\theta))\left(d_{1}-70\right)}{128(\mathrm{ab-}-23.75)^{2}}\right)\right) \ddot{d}_{1}$
where

$$
\begin{aligned}
& I 1=\left(l_{1}-l_{2}\right)^{2}-\left(l_{3}-l_{4}\right)^{2}-140\left(l_{1}-l_{2}\right)+140\left(l_{3}-l_{4}\right) \\
& I 2=\left(l_{1}-l_{2}\right)\left(\dot{l}_{1}-\dot{l}_{2}\right)-\left(l_{3}-l_{4}\right)\left(\dot{l}_{3}-\dot{l}_{4}\right)-70\left(\dot{l}_{1}-\dot{l}_{2}\right)+70\left(\dot{l}_{3}-\dot{l}_{4}\right) \\
& \sin (\theta)=\frac{\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{4 c} \\
& \cos (\theta)=\sqrt{1-\left(\frac{\left.l_{1}+l_{2}-l_{3}-l_{4}\right)}{4 c}\right)^{2}} \\
& d_{1}=l_{1}-l_{2} \\
& d_{2}=l_{3}-l_{4} \\
& \ddot{z}_{c}=\frac{\dot{B} \dot{A}-A \ddot{B}}{2 A^{2}}+\frac{\ddot{O} \sqrt{B^{2}-4 A O}-\dot{C}[B \dot{B}-2(A \dot{C}+C \dot{A})] \sqrt{B^{2}-4 A C}}{B^{2}-4 A C} \\
& -\frac{A \sqrt{B^{2}-4 A C}\left[B \ddot{B}+\dot{B}^{2}\right]-B \dot{B}\left[\dot{A} \sqrt{B^{2}-4 A C}+A\left(B^{2}-4 A C\right)^{-\frac{1}{2}}\{B \dot{B}-2(A \dot{C}+C \dot{A})\}\right]}{2 A^{2}\left(B^{2}-4 A C\right)} \\
& +\frac{A \sqrt{B^{2}-4 A C}[\dot{A} \dot{C}+C \ddot{A}]-\dot{A} C\left[\dot{A} \sqrt{B^{2}-4 A C}+A\left(B^{2}-4 A C\right)^{-\frac{1}{2}}\{B \dot{B}-2(A \dot{C}+C \dot{A})\}\right]}{A^{2}\left(B^{2}-4 A C\right)} \\
& +\frac{A\left[\ddot{A} \sqrt{B^{2}-4 A C}+\dot{A}\left(B^{2}-4 A C\right)^{-\frac{1}{2}}\{B \dot{B}-2(A \dot{C}+C \dot{A})\}\right]-2 \dot{A}^{2} \sqrt{B^{2}-4 A C}}{2 A^{3}} \\
& +\frac{A[\dot{A} \dot{B}+B \ddot{a}]-2 \dot{A}^{2} B}{2 A^{3}} \\
& \ddot{z}_{M H c m}=\ddot{z}_{c}+E c \frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}}{16 c^{2} \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}} \\
& +E c \frac{\left(\ddot{l}_{1}+\ddot{l}_{2}-\ddot{l}_{3}-\ddot{l}_{4}\right)}{\sqrt{16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}}} \\
& -E c \frac{\left(l_{3}+l_{4}-l_{1}-l_{2}\right)\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)^{2}}{\left[16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}\right]^{\frac{3}{2}}} \\
& \ddot{y}_{\text {MHcm }}=\frac{\left(l_{1}-l_{2}\right)\left(\ddot{l}_{3}-\ddot{l}_{2}\right)+\left(\dot{l}_{1}-\dot{l}_{2}\right)^{2}-\left(l_{3}-l_{4}\right)\left(\ddot{l}_{3}-\ddot{l}_{4}\right)-\left(\dot{l}_{3}-\dot{l}_{4}\right)^{2}-70\left(\ddot{l}_{1}-\ddot{l}_{2}-\ddot{l}_{3}+\ddot{l}_{4}\right)}{\delta(a-b-23.75)} \\
& \frac{-\left(d-c \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}\right) \ddot{z}_{c}}{(a-b-23.75)} \\
& -\frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-\dot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{8 c(a-b-23.75) \sqrt{1-\left(\frac{l_{2}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}} \dot{z}_{c}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\left(\ddot{l}_{1}+\ddot{l}_{2}-\ddot{l}_{3}-\ddot{l}_{4}\right)\left(l_{1}+l_{2}-l_{3}-l_{4}\right)}{4(a-b-23.75) \sqrt{16^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}}}\left(z_{c}+104\right) \\
& \frac{\left(l_{1}+l_{2}-l_{3}-l_{4}\right)^{2}\left(\dot{l}_{1}+\dot{i}_{2}-\dot{l}_{3}-i_{4}^{4}\right)^{2}}{4(a-b-23.75)\left[16 c^{2}-\left(l_{3}+l_{4}-l_{1}-l_{2}\right)^{2}\right]^{\frac{3}{2}}}\left(z_{c}+104\right) \\
& \frac{\left(\dot{l}_{1}+\dot{l}_{2}-\dot{l}_{3}-i_{4}\right)^{2}}{16 c(a-b-23.75) \sqrt{1-\left(\frac{l_{1}+l_{2}-l_{3}-l_{4}}{4 c}\right)^{2}}}\left(z_{c}+104\right)
\end{aligned}
$$

For the platform component as shown in equations (12) and (12a)-(12c), we need to obtain the derivative with respect to $l_{i}$ of the first and second time derivative obtained above by applying chain rule. Then, rearranging the expressions, we will get translational force on each joint due to mass and inertia of milling head. We continue to do the same procedure on the remaining components as shown in the equations (13), (14), (15) and (16). Finally, total force ( $F_{i}$ ) can be obtained by summing all forces which are generated from masses and moments of inertia of other components.

## 5. Simulation Results

The dynamic model obtained from previous section is implemented in MATLAB in order to compare the numerical solutions with the commercial dynamics software such as the ADAMS solver.

The model parameters used in the simulation are $\mathrm{Tl}=$ $163 \mathrm{~mm}, \mathrm{c}=37.5 \mathrm{~mm}, \mathrm{R}=587 \mathrm{~mm}, \mathrm{~b}=57.25 \mathrm{~mm}, \mathrm{a}=$ $215 \mathrm{~mm}, \mathrm{~h}=1024.5 \mathrm{~mm}, \mathrm{p}=720 \mathrm{~mm}, \mathrm{~d}=37.5 \mathrm{~mm}, \mathrm{Ec}=$ 22.93 mm . We use materials as mild steel for all components. So, mass properties and moments of inertia can be found as follows:

$$
\begin{aligned}
& m_{M H}=14.0881 \mathrm{~kg} \\
& m_{C H}=9.36853 \mathrm{~kg} \\
& m_{A m}=2.7194 \mathrm{~kg} \\
& m_{L J}=4.65962 \mathrm{~kg} \\
& m_{U C}=0.28739 \mathrm{~kg} \\
& \bar{I}_{M H y}=94625.076 \mathrm{kgmm}^{2} \\
& \bar{I}_{X X, A m}=105660 \mathrm{kgmm}^{2} \\
& \bar{I}_{Y Y, A m}=105415 \mathrm{kgmm}^{2} \\
& \bar{I}_{U C y}=60.3645 \mathrm{kgmm}^{2}
\end{aligned}
$$

Figure 9 shows the circular path on a spherical surface workpiece. The path is used for the evaluation of the dynamic model. The diameter of the circular path is 104.1889 mm . In the simulation, the tool-tip is always point toward to the direction normal to the plane which tangent to the spherical surface along the circular path. Knowing the position and orientation of points on the circular path as shown in Figure 9, the variables $\mathrm{q}=\left[\begin{array}{lllll}l_{1} & l_{2} & l_{3} & l_{4} & \alpha\end{array}\right]^{T}$ together with $\theta$ (rotating table) can be solved by inverse kinematics equations. The feed rate along the path is equal to $27.25 \mathrm{~mm} / \mathrm{sec}$ or 1635.45 $\mathrm{mm} / \mathrm{min}$. The four translational forces exert on the two
rails of the $\mathrm{H}-4$ mechanism as the tool-tip move along the path on the spherical surface are obtained from both MATLAB and ADAMS Solver. The GSTIFF algorithm of the ADAMS solver is used to obtain the numerical solution from the solid modeling of the manipulator arm with 24 seconds simulation time and 0.06 second for time increment. For MATLAB, the numerical solution obtained from the derived dynamic model. The four translational forces from both simulations are shown in Figure 11. And the errors between both simulations are shown in Figure 10. The errors in the translational forces are within 0.0006 Newton. This implied that the derived dynamic model is accurate enough from using in the feedback control algorithm


Figure 9. Circular path on spherical workpiece


Figure 10. Errors in Translational Force (N) of $l_{1}, l_{2}, l_{3}, l_{4}$ between numerical solution of the analytical equation and the results from ADAMS


Figure 11. Translational Generating Force of joint l1-l4

## 6. Experimental Results

According to dynamic equations derived in previous section, we setup an experiment to compare the joint torques from derived model (inverse dynamic) with the measured joint torques. Inverse kinematics is used for obtaining the joint commands from the desired circular working paths as shown in Figure 9. The inputs to the system are the desired trajectories, described as function of time as $\quad \mathbf{x}_{T}^{T P}(t)=\left[\begin{array}{llll}x_{w}(t) & y_{w}(t) & z_{w}(t) & \theta(t)\end{array} \alpha(t)\right]^{T}$. The outputs are the joint torques to be applied at each instant by the brushless DC servo motors in order to follow the desired circular trajectories path. The desired actuated torque can be obtained by evaluating the right-hand side terms of the dynamic equations as following:

$$
\begin{gather*}
F_{i}=\frac{d}{d t} \frac{\partial(K . E .)}{\partial \dot{l}_{i}}-\frac{\partial(K . E .)}{\partial l_{i}}+\frac{\partial(P E .)}{\partial l_{i}} \quad i=1,2,3,4 \\
\text { and } \\
\tau_{i}=\frac{\mathrm{F}_{\mathrm{i}} \mathrm{D}_{\mathrm{p}}}{2}\left[\frac{L_{d}+\mu \pi \mathrm{D}_{\mathrm{p}}}{\pi \mathrm{D}_{\mathrm{p}}-\mu L_{d}}\right]+\left(J_{\text {motor }}+J_{\text {sorew }}\right)\left(\frac{2 \pi \ddot{l}_{i}}{L_{d}}\right)+\tau_{\text {friction } 1} \tag{18}
\end{gather*}
$$

The torque in equation (18) is derived from the assumption that the platform is moving upward. The ball screws used have self-locking characteristic. So, the force need to overcome the gravitational force distributes in the direction of the rail. For the opposite direction, the torque require to move the platform is

$$
\begin{equation*}
\tau_{i}=\frac{\mathrm{F}_{\mathrm{i}} \mathrm{D}_{\mathrm{p}}}{2}\left[\frac{\mu \pi \mathrm{D}_{\mathrm{p}}-L_{d}}{\pi \mathrm{D}_{\mathrm{p}}+\mu L_{d}}\right]+\left(J_{\text {motor }}+J_{\text {oorew }}\right)\left(\frac{2 \pi \ddot{l}_{i}}{L_{d}}\right)-\tau_{\text {friction } 2} \tag{19}
\end{equation*}
$$

$\mathrm{D}_{\mathrm{p}}=$ pitch diameter of ball screw $=0.016 \mathrm{~m}$
$\mathrm{L}_{\mathrm{d}}=$ lead of the screw in mm per revolution $=0.005 \mathrm{~m}$
$\mu=$ coefficient of friction for rolling contact $=0.01687$
$\mathrm{J}_{\text {motor }}+\mathrm{J}_{\text {screw }}=$ Inertia of rotor and Ball screw

$$
\begin{aligned}
& =0.0000756877 \mathrm{~kg}-\mathrm{m}^{2} \text { for motor } 1,2,3 \\
& =0.0001576877 \mathrm{~kg}-\mathrm{m}^{2} \text { for motor } 4
\end{aligned}
$$

$\tau_{\text {friction }}=$ overall resistant torque due to coulomb friction of all joints

The frictions torque can be obtained from experiment by measure the force or torque to move the rail on the ball screw. The friction torques are not constant, they are
function of angle $\Theta$. The linear regression model of the friction torque can be obtained as following:

For motor1:

$$
\begin{align*}
& t_{\text {friction } 1}=0.0019743 * q+0.205887  \tag{20}\\
& t_{\text {friction } 2}=-0.0022429 * q+0.112949 \tag{21}
\end{align*}
$$

For motor2:

$$
\begin{align*}
& t_{\text {friction } 1}=-0.00112815 * q+0.2143485  \tag{22}\\
& t_{\text {friction } 2}=-0.00338445^{*} q+0.14665915 \tag{23}
\end{align*}
$$

For motor3:

$$
\begin{align*}
& t_{\text {friction } 1}=-0.0022563 * q+0.179193  \tag{24}\\
& t_{\text {friction } 2}=0.00112815 * q+0.0776595 \tag{25}
\end{align*}
$$

For motor4:

$$
\begin{align*}
t_{\text {friction } 1} & =-0.000112815 * q+0.4174155  \tag{26}\\
t_{\text {friction } 2} & =0.00112815^{*} q+0.3046005 \tag{27}
\end{align*}
$$

while the angle $\theta$ is the orientation of tooltip in unit of degree

The torques evaluate from the equation (17)-(19) can be compared with the actuated torques measured from the experiment. And the results of the comparison are shown in Figure 12. The average torque of each motor, from the measurement, is very close to the torque obtained from the derived dynamic model.




Figure 12. Compare torques between simulation model by derived equation and experiment.

## 7. Conclusion

In this paper, we presented the kinematics analysis, Jacobian and the Dynamic Model of the 5-DOF parallel mechanism design base on H-4 family. The forward and inverse kinematics analysis are presented. Singularities analysis is shown only for the H-4 type. The dynamic equations are derived from the Lagrangian formulation. The numerical results of the analysis of kinematics, Jacobian and the dynamic model are compared with numerical results from a popular commercial software, the ADAMS solver, using the solid model data. This is to assure the accuracy the derived model and possibility of using it in the feedback system. Friction models obtained from the experiment are used to compensate the actual friction of the system in the resolve acceleration control strategy.

## References

[1]. Monsarrat, B., Gosselin, C. M. Singularity Analysis of a Three-Leg 6Dof parallel Grassmann Line Geometry. International Journal of Robotics Research vol. 20 no.4, April 2001, pp. 312-326.
[2]. Tsai, L. W. Robot Analysis-The Mechanics of Serial and Parallel Manipulators. John Wiley \& Sons, 1999.
[3]. Clavel, R., "Conception d'un robot parallèle rapide à 4 degrés de liberté," Ph.D. Thesis, EPFL, Lausanne, Switzerland, 1991.
[4]. Pierrot, F. H4_a new family of 4-DOF parallel robots. IEEE/ASME Advanced Intelligent

Mechatronics Conf. Proc. Atlanta USA, September 1999, pp. 508-513.
[5]. Gosselin, C. M., Angeles, J. Singularity analysis of closed-loop kinematics chains. IEEE Transactions on Robotics and Automation vol. 6 no. 3, June, 1990. pp. 281-290.
[6]. Sangveraphunsiri,V. and Tantawiroon N., Novel Design of a 4 DOF parallel Robot., 2003 JSAE Annual Congress, Yokohama, Japan, May21-23, 2003.
[7]. Dumitru Olaru, George C. Puiu, Liviu C. Balan, Vasile Puiu, A New Model to Estimate Friction Torque in a Ball Screw System. Product Engineering Springer Netherlands, page333346,2005
[8]. Viboon Sangveraphunsiri and Prasartporn Wongkumchang, Design and Control of a Stewart Platform, the $15^{\text {th }}$ National Conference of Mechanical Engineering,2001, (in Thai).
[9]. Viboon Sangveraphunsiri and Tawee Ngamvilaikorn, Design and Analysis of 6 DOF Haptic Device for Teleoperation Using a Singularity-Free Parellel Mechanism, Thammasat International Journal of Science and Technology vol. 10 No. 4 October-December 2005,pp. 60-69.

