Applications of Class-Gamma Stability Analysis Theorem for Robust Stability and Performance via Gain Scheduling

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Abstract

This paper presents a new approach for applying robust stability analysis theorems to improve response characteristics of control systems by general gain scheduling. Using the proposed design procedure, allowable bounds for stability of scheduled gains may be obtained for all scheduling schemes being continuous differentiable and globally Lipschitz in relevant scheduling variables. We apply the procedure to obtain a PID controller with significantly large scheduling bounds on the integral gain for joint control of a modified SCARA robot arm. The procedure could yield not only input-to-state stability for typical uncertain nonlinear systems but also possibility to improve performance by means of gain scheduling.

Keywords

Stability, Class Gamma, PID, Gain Scheduling, Robot

I. Introduction

The problem of robust stability analysis (RSA), in which allowable bounds on nonlinear time-varying uncertainties (NTU) are computed for stable linear control systems, has been considered in parallel with the problem of robust controller design [1-4]. While it seems that no major result on the problem of robust quadratic stabilization has been reported after the publication of [5], robust stability analysis theorems have been steadily formulated to reduce conservatism of allowable uncertainty bounds. These useful RSA theorems have been accumulated for many years, and they now cover various classes of uncertainties. Motivated by this fact, [6] employed matrix algebra and geometry to propose a new class-gamma RSA theorem, and a technique for extending the uses of all RSA theorems in class gamma over robust controller design of single-input linear systems with NTU. Using these, it was shown in numerical examples that the resulting allowable uncertainty bounds could be less conservative than those resulting from [5]. The number of inputs is allowed to be multiple in [7]. This paper extends applications of RSA theorems to address both the problems of stability and performance in the same setup. The extension is achieved by strengthening class gamma theorems in [6, 7] to guarantee exponential stability rather than asymptotic stability, while casting scheduled variations of state feedback gains as psudo-uncertainty. The proposed setup is not specific to a particular gain scheduling scheme. It requires only that the scheme is continuously differentiable in its variables and is globally Lipschitz.

2. Mathematical Description

In this paper, we are interested in computing linear control laws that guarantee input-to-state stability for the perturbed linear systems with nonlinear time-varying uncertainties:

$$\dot{x} = [A + \Delta A(x,t)]x - [B + \Delta B(x,t)][K + \Delta K(x,t)]x + f(x,u) \tag{1}$$

where $x \in \Re^n$ is the state vector, the system matrix $A \in \Re^{n\times n}$ is known, the input matrix $B \in \Re^{n\times m}$ is known, $K \in \Re^{m\times n}$ is the nominal state feedback gain matrix to compute, the state-independent input $u \in \Re^p$ is unknown, the bounded nonlinear uncertain perturbation $f(x,u) \in \Re^n$ is unknown, and Δ denotes time-varying nonlinear uncertainties with appropriate dimensions and known bounds for matrices A and B. Special attention should be drawn towards $\Delta K(x,t)$ which represents scheduling variations in the effective feedback gain matrix $K + \Delta K(x,t)$ of the required gain scheduling system. The origin is the equilibrium point of Eq.(1), whose right hand side is continuously differentiable and is globally Lipschitz in x and u, uniformly in t. It can be shown using Lyapunov stability theorem that the system is input-to-state stable, that is bounded inputs produce bounded states, if the equilibrium point at the origin of the unperturbed system is globally exponentially stable [8]. Accordingly, it remains to show in this paper that the unperturbed system has the required property when subjected to time-varying nonlinear uncertainties. This is given in the next section.

3. A Strengthened Class Gamma Theorem

Given a nominal gain matrix K and all associated uncertainty specifications, it is always possible to write the unperturbed system as proposed in [6, 7]:

$$\dot{x} = Ax - BKx + \sum_{j=1}^{r}[h_j(x,t)E_j]x = \bar{A}x + \sum_{j=1}^{r}[h_j(x,t)E_j]x \quad (2)$$

where $\bar{A} \equiv A - BK$ is known, $E_j \in \Re^{n\times n}$ is known, and $h_j(x,t) \in [h_{jl},\ h_{uj}]$ is a time-varying nonlinear uncertain function with known bounds. References [6, 7] proposed class-gamma theorems that guarantee asymptotic stability of the unperturbed system when written in the form of Eq.(2) and certain computational requirements are satisfied. However, this type of stability is insufficient to guarantee input-to-state stability for the perturbed system. We show in this paper that the requirements in these class-gamma theorems are indeed sufficient to exert exponential stability in the large for the unperturbed system by incorporating additional arguments into the corresponding proofs.

Theorem 1 If the dynamical system in Eq.(2) is continuously differentiable and is globally Lipschitz with matrix $\bar{A}$ being Hurwitz and

$$\max(\lambda(Z)) < 0 \quad (3)$$

then the equilibrium point at the origin is globally exponentially stable. The matrix $Z = Z^T \in \Re^{n\times n}$ is obtained by:

1) Specified Q > 0 and $\bar{A}$ to compute P from the Lyapunov equation.
2) Compute $\bar{A}_l = \bar{A} + \sum_{j=1}^{r} h_{lj}E_j$, and $\Phi = P\bar{A}_l + \bar{A}_l^T P$.
3) Compute $\Psi_j = [PE_j + E_j^T P] = \Psi_j^T$.
4) Compute $\Lambda_{\Psi_j} = T_{\Psi_j}^T \Psi_j T_{\Psi_j} = \mathrm{diag}[\lambda_{\Psi_{j1}}\ \ \cdots\ \ \lambda_{\Psi_{jn}}]$, where $T_{\Psi_j} = [\,v_{\Psi_{j1}}\ |\cdots|\ v_{\Psi_{jn}}]$, and $\{v_{\Psi_{j1}},\ \ldots\ ,v_{\Psi_{jn}}\}$ is the set of *n* orthonormal eigenvectors of Ψ_j.
5) Compute $\Lambda_{\Psi_j}^{\geq 0}$ by setting all negative elements of Λ_{Ψ_j} to zero
6) Compute $\Psi_j^{\geq 0} = T_{\Psi_j}\Lambda_{\Psi_j}^{\geq 0}T_{\Psi_j}^T$.
7) Compute $Z \equiv \Phi + \sum_{j=1}^{r}[(h_{uj} - h_{lj})\Psi_j^{\geq 0}]$.

Proof We write for $h_j(x,t)$, j = 1, 2, ..., r:

$$h_j(x,t) = h_{lj} + h_j(x,t) - h_{lj} \equiv h_{lj} + l_j(x,t) \quad (4)$$

where $l_j(x,t) \equiv h_j(x,t) - h_{lj}$. Since $h_j(x,t) \in [h_{lj},\ h_{uj}]$, $l_j(x) \in [0,\ h_{uj} - h_{lj}]\forall j$. Substituting $h_{lj} + l_j(x)$ for $h_j(x,t)$ in Eq.(2) yields:

$$\dot{x} = \bar{A}_l x + \sum_{j=1}^{r} l_j(x,t)E_j x \quad (5)$$

Now put Q > 0 into the Lyapunov equation $-Q = (1/2)[P\bar{A} + \bar{A}^T P]$, solve for P, and obtain $\Phi = P\bar{A}_l + \bar{A}_l^T P$. Note that the Lyapunov function V(x)=$(1/2)x^T Px$ is such that $P = P^T > 0$ and

$$(1/2)\min(\lambda(P))\|x\|^2 \leq V(x) \leq (1/2)\max(\lambda(P))\|x\|^2$$

Accordingly, $V(x)$ is bounded from above and below by class K functions. Differentiating V(x) along trajectories of Eq. (5) yields:

$$\dot{V}(x,t) = (1/2)x^T\Phi x + (1/2)\sum_{j=1}^{r} l_j(x,t)x^T\Psi_j x \quad (6)$$

Since $\Psi_j^T = \Psi_j\ \ \forall j$, Ψ_j has a set of n real eigenvalues $\{\lambda_{\Psi_{j1}},\ \ldots\ ,\lambda_{\Psi_{jn}}\}$ and the corresponding set of n orthonormal eigenvectors $\{v_{\Psi_{j1}},\ \ldots\ ,v_{\Psi_{jn}}\}$. Using the linear transformation $x = T_{\Psi_j}z$, we write:

$$x^T\Psi_j x = z^T[T_{\Psi_j}^T\Psi_j T_{\Psi_j}]z \equiv z^T\Lambda_{\Psi_j}z \quad (7)$$

with $T_{\Psi_j} = [\,v_{\Psi_{j1}}\ |\cdots|\ v_{\Psi_{jn}}]$, $\dot{x} = \bar{A}_l x + \sum_{j=1}^{r} l_j(x,t)E_j x$. We set all negative elements of $\Lambda_{\Psi j}$ to zeros to produce $\Lambda_{\Psi j}^{\geq 0}$. Thus, $z^T[\Lambda_{\Psi_j}^{\geq 0}]z \geq 0$, and

$$z^T[\Lambda_{\Psi j}^{\geq 0}]z \geq z^T\Lambda_{\Psi j}z = x^T\Psi_j x$$

It follows that:

$$z^T[\Lambda_{\Psi_j}^{\geq 0}]z = x^T[T_{\Psi_j}^{-1}]^T[\Lambda_{\Psi_j}^{\geq 0}][T_{\Psi_j}^{-1}]x \equiv x^T\Psi_j^{\geq 0}x \geq 0 \quad (8)$$

where $\Psi_j^{\geq 0} = [T_{\Psi_j}^{-1}]^T[\Lambda_{\Psi_j}^{\geq 0}][T_{\Psi_j}^{-1}]$.

Because T_{Ψ_j} is orthogonal, we have $T_{\Psi_j}^{-1} = T_{\Psi_j}^T$, and $\Psi_j^{\geq 0} = [T_{\Psi_j}][\Lambda_{\Psi_j}^{\geq 0}][T_{\Psi_j}^T]$. Now, because $[\Psi_j^{\geq 0}]^T = \Psi_j^{\geq 0}$ and because $(h_{uj} - h_{lj}) \geq l_j(x) > 0$, it follows that:

$$l_j(x,t)[x^T\Psi_j x] \leq (h_{uj} - h_{lj})[x^T\Psi_j^{\geq 0}x]\ \forall x \quad (9)$$

Applying the above inequality to Eq.(6) yields:

$$\dot{V}(x,t) \leq \frac{1}{2}x^T\Phi x + \frac{1}{2}\sum_{j=1}^{r}((h_{uj} - h_{lj})[x^T\Psi_j^{\geq 0}x]) \quad (10)$$

Now, by letting $Z \equiv \Phi + \sum_{j=1}^{r}[(h_{uj} - h_{lj})\Psi_j^{\geq 0}]$, we have $\dot{V}(x,t) \leq (1/2)x^T Zx$. If $\max(\lambda(Z)) < 0$, then

$$\dot{V}(x,t) \leq -(1/2)|\max(\lambda(Z))|\|x\|^2 \quad (11)$$

which indicates that time derivative of the Lyapunov function is bounded from above by negative of a class K function. Accordingly, the origin of the unperturbed linear system with time-varying nonlinear uncertainties (2) is a globally exponentially stable equilibrium [8]. This completes the proof. ⊗

This class gamma theorem determines if a nominal linear state feedback gain matrix K yields exponential stability in the large for the equilibrium point at the origin of the interested system. It does not provide such stabilizing gain matrix directly. Accordingly, a technique for generating reasonable stabilizing candidates is required. This is addressed in the next section.

4. Controller Design Procedure

It has been shown in [6, 7] that if the nominal linear state feedback gain matrix is generated in a certain fashion, then relative orientation of null surfaces associated with the time derivative of the quadratic Lyapunov function along trajectory of a nominal linear system has a certain symmetry property. In addition, this property offers stability robustness for the nominal linear system when subjected to time-varying nonlinear uncertainties. This controller generation technique is incorporated as a part of the controller design procedure in [6, 7] for stabilizing a class of linear systems with time-varying nonlinear uncertainties therein. Here, with extended objective to find stabilizing gain matrix with allowable gain scheduling bounds that guarantees a stronger type of stability, we find in realistic problems that this controller design procedure remains effective. The procedure is given here briefly for convenience of references.

Procedure

1) Define a two dimensional domain of $\rho > 0$ and $\eta \geq 1$, and select a grid size for this domain.
2) Select coordinate $(\rho,\ \eta)$, then complete step 2 – 5.
3) solve for P from $-2I = PA + A^T P - 2\rho PBB^T P$.
4) Compute K from $K = \eta\rho B^T P$.
5) Compute Q from $Q = I + (\eta - 1)N$.
6) For each pair of (K, Q) obtain in the previous steps, compute $\max(\lambda(Z))$ in Theorem 1. If Theorem 1 is satisfied, then K is a stabilizing solution. The procedure may stop here, or may loop through steps 2 – 6 to find other stabilizing solutions that may be more suitable to the system in some way.

This procedure can be automated in numerical packages such as MATLAB, and yields stabilizing solutions for many realistic problems in seconds.

5. Example

Consider the problem in which a DC actuating motor for joint 2 of a modified SCARA robot depicted in Fig. 1 is controlled to track a time-varying trajectory. Because the electrical time constant of a DC joint motor is significantly less than the corresponding mechanical time constant, motor dynamics can be represented by a reduced-order mathematical model:

$$J_m\ddot{\theta} + (B_m + \frac{K_{bm}K_m}{R_m})\dot{\theta} + r_T T_d = \frac{K_m}{R_m}V$$

where J_m is the mass moment of inertia of the rotor, B_m is the viscous damping coefficient, K_{bm} is the back-EMF constant, K_m is the torque constant, R_m is the resistance of the armature, r_T is the joint mechanical transmission ratio, T_d is the loading torque resulting from coupled manipulator dynamics, V is the motor driving voltage, and θ is the rotational angle of the motor in radian. For this joint, examining equation of motion of the manipulator reveals that the linear dynamics of the motor is perturbed by torque resulting robot operation:

$$T_d = \left(m_2 L_{C2}^2 + m_2 L_1 L_{C2}\cos(q_2) + \bar{I}_{Z2}\right)\ddot{q}_1$$
$$+\left(m_2 L_{C2}^2 + \bar{I}_{Z2}\right)\ddot{q}_2 + m_2 L_1 L_{C2}\sin(q_2)\dot{q}_1^2$$

where L_1 is the length of link 1, measured from rotational axe of link 1 to that of link 2, L_{C2} is the length from rotational axis of link 2 to the center of gravity of the same link, m_2 is the mass of link 2 assembly, including mass of joint motor 2, $\bar{I}_{Z2}$ is the mass moment of inertia of link 2 about the rotational axis, q_i is the rotational angle of link i where i = 1, 2. Relevant physical parameter for this SCARA robot is given in Table 1. Some of these are obtained by direct measurement and some by computations. The leasts of them are listed in Table 1, with the corresponding errors expected to be no larger than +5%.

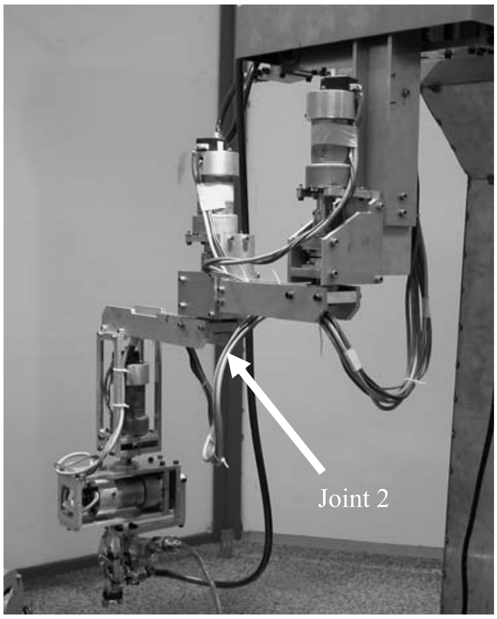

Fig. 1 The modified SCARA robot in the example

Now, define the error signal:

$$e = r - \theta$$

where r is the referenced signal for θ. It follows that $\dot{e} = \dot{r} - \dot{\theta}$, and $\ddot{e} = \ddot{r} - \ddot{\theta}$. We agree that r and its time derivatives are continuously differentiable, and that errors associated with measurements on r_T, $\dot{r}$, $\ddot{r}$, $\dot{q}$, and $\ddot{q}$ are negligible. Substituting $\theta = r - e$ and its time derivatives in the simplified equation yields:

$$J_{eff}\ddot{e} + B_{eff}\dot{e} = -G_{eff}V + F_{ueff}(q_1, q_2, \dot{r}, \ddot{r})$$

where $G_{eff} = K_m / R_m$, $J_{eff} \equiv J_m + \left(m_2 L_{C2}^2 + \bar{I}_{Z2}\right) r_T^2$, $B_{eff} = B_m + (K_{bm}K_m / R_m)$, and $F_{ueff}(q_1, q_2, \dot{r}, \ddot{r}) = J_{eff}\ddot{r} + B_{eff}\dot{r} + r_T T_d$.

Letting $\int e\ dt = x_1$, $e = x_2$, and $\dot{e} = x_3$, we write the perturbed uncertain system in the vector-matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -(3.8+h_{A1}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -(89.3+h_{B1}) \end{bmatrix} V + \begin{bmatrix} 0 \\ 0 \\ F_{ueff}(u)/J_{eff} \end{bmatrix}$$

where $h_{A1} \in [0,\ 0.86] \equiv [h_{l,A1},\ h_{u,A1}]$, $h_{B1} \in [0,\ 20.52]$ $\equiv [h_{l,B1},\ h_{u,B1}]$, and $u = [q_1 \quad q_2 \quad \dot{r} \quad \ddot{r}]^T$. Now, let us employ the PID gain scheduling control law:

$$V = -K(x)x$$

where $K(x) = \left[K_i + \Delta K_i(x) \quad K_p \quad K_d\right]$, K_i, K_p and K_d are constants, and $\Delta K_i(x)$ is scheduled change of the integral gain K_i according to the error signal and its first time derivative. The scheduling scheme is imposed to be continuously differentiable and is globally Lipschitz in the scheduled variables. It then follows that the right hand side of the equation of motion is continuously differentiable, and is globally Lipschitz in x and u.

Sym.	Description	Value	Unit
J_m	Mass moment of inertia of rotor	1.62×10^{-5}	kg.m^2
B_m	Viscous damping coefficient	1.15×10^{-5}	N.m.s
K_{bm}	Back-EMF constant	1/26	V.s
K_m	Torque constant	1/26	N.m/A
R_m	Resistance of the armature	7.3	Ohm
r_T	Joint mechanical transmission ratio	1/78	-
L_1	Length of link 1, measured from rotational axe of link 1 to that of link 2	0.5	m
L_{C2}	Length from rotational axis of link 2 to the center of gravity of the same link	0.2	m
m_2	Mass of link 2 assembly, including mass of joint motor 2	4.65	kg
$\bar{I}_{Z2}$	Mass moment of inertia of link 2 about the rotational axis	0.0218	kg.m^2

Table 1 Physical parameters for the modified SCARA robot in example

The equation of motion represents a linear system with nonlinear uncertainties in parameters and state-dependent perturbation $F_{ueff}(q_1, q_2, \dot{r}, \ddot{r})/J_{eff}$. Because of these undesirables, we do not expect asymptotic stability for the autonomous system. Rather, we pursue a more realistic input-to-state stability in which bounded inputs produce bounded errors. To obtain this, it remains to show exponential stability of the unperturbed system about the equilibrium point at the origin. The dynamics of the unperturbed scheduled uncertain system is now given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_{A2}(x) & 0 & -(3.8+h_{A1}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -(89.3+h_{B1}) \end{bmatrix} K_n$$

where, $h_{A2}(x) \equiv -(89.3+h_{B1})\Delta K_i(x)$ and $K_n = \left[K_i \quad K_p \quad K_d\right]$ is the nominal gain matrix. Note that we have accounted $\Delta K_i(x)$, the scheduled variation of the integral gain, as an additional uncertain element $h_{A2}(x)$ of the system matrix. Using the controller generating procedure in [6-7], we obtain a stabilizing solution:

$$K_n = [-2.53 \quad -4.4 \quad -2.48]$$

Using Theorem 1, it can be shown that the above solution guarantees exponential stability for the unperturbed scheduling uncertain system and all the integral gain scheduling schemes, provided that the integral gain variation $\Delta K_i(x)$ is scheduled to be within $\pm(0.56\times100/2.53)\% = \pm 22.1\%$ of its nominal value of -2.53. The perturbed system is then input-to-state stable under this condition. For computation references, we give relevant matrices resulting from the application of Theorem 1:

$$P = \begin{bmatrix} 3.48 & 2.03 & 0.018 \\ 2.03 & 3.52 & 0.03 \\ 0.018 & 0.03 & 0.017 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 5.22 & 2.95 \\ 5.22 & 10.09 & 5.13 \\ 2.95 & 5.13 & 3.89 \end{bmatrix},$$

and $Z = \begin{bmatrix} -5.07 & -9.59 & -5.42 \\ -9.59 & -18.7 & -9.43 \\ -5.42 & -9.43 & -7.31 \end{bmatrix}$.

The eigenvalues of Z are -0.0195, -1.99, and -29.08. To see system responses in a typical situation, we let the integral gain be scheduled within the allowable bounds by a fuzzy logic scheduler. All the membership functions are Gaussian and Sigmoid, which are continuously differentiable and is globally Lipschitz. Details on the scheduler are not the main objective of this paper and thus will not be discussed. We let r = (t + 0.1sin(t))/r_T, $\dot{q}_1 = 0.2\sin(t)\cos(t)$, and all the relevant physical parameters are at their minimums. The resulting response is given in Fig. 2.

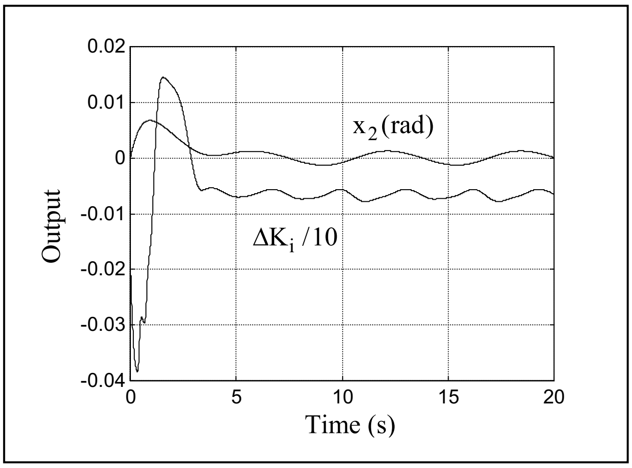

Fig. 2 Response of the integral gain scheduling PID system

6. Conclusion

In this paper, previous applications of class-gamma theorems on controller design for robust stability of perturbed linear systems with time-varying nonlinear uncertainties have been extended to provide robust performance by means of gain scheduling. This extension is centered around a strengthened class-gamma theorem that guarantees in the large exponential stability, rather than asymptotic stability, for the unperturbed system. By treating the required gain scheduling variations as psudo-uncertainties in a special fashion, the class-gamma theorem could guarantee input-to-state stability for the uncertain system when subjected to state-dependent perturbations. It yields solutions for both stabilizing gain matrix and gain scheduling bounds at the same time. Except the required conditions of being continuously differentiable and globally Lipschitz, our controller design procedure does not assume a specific scheduling scheme to come up with solutions. According, a gain scheduling scheme can be selected thereafter to be most effective for different problems.

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