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Linear Dynamic Systems Optimization: A Fourth-Order TPBVP Formulation with Weighted Residual

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Abstract

This paper deals with dynamic optimization of vibrating spring-mass-damper systems which are characterized by n second-order linear coupled differential equations with constant coefficients, where n is the number of degrees-of-freedom of the system. The system is commanded to move between two fixed states in a prescribed final time. In the literature, the optimal solution of this problem is obtained by simultaneously solving $4n$ first-order linear differential equations in the state and costate variables. The boundary conditions are stated in terms of $2n$ state variables at the initial and final time. As a result, shooting methods are used with guesses on the costate variables until a desired convergence is achieved. The solution procedure is highly computer intensive and the convergence is sensitive to initial guess.

In this paper, a different form of the variational statement is presented. As a consequence, instead of $4n$ first-order differential equations, the system is described by n fourth-order differential equations in the state variables. Since the costate variables do not appear explicitly in the optimality equations, the problem becomes highly attractive to solve using classical weighted residual methods, such as Galerkin scheme. Some salient features of this scheme are: (i) convergence is achieved with a very small number of modes, (ii) the solution of state and control trajectories is not iterative and requires inversion of *only one* $(mn \times mn)$ matrix, where n is the number of degrees-of-freedom in the system and m is the number of modes chosen in the analysis.

The authors believe that the solution of fourth-order TPBVP with weighted residual methods has tremendous computational benefits over shooting point methods for real-time control applications.

1. Introduction

The dynamic model of an n degree-of-freedom mechanical system consisting of spring, masses, and dampers can be written in the following general form:

$$M\ddot{X} + C\dot{X} + KX = U \tag{1}$$

where M, C, and K are $(n \times n)$ symmetric matrices usually referred to as the mass, damping, and stiffness matrices, respectively [6]. Also, M is a positive definite matrix. X is an n-dimensional vector of generalized coordinates. U is an n-dimensional vector of actuator inputs and 'dots' denote the derivatives of variables with respect to time.

The problem statement is to choose the control-input vector $U(t)$ that will take the system from an initial position X_0 and initial velocity $\dot{X}_0$ at time t_0 to a final position X_f and final velocity $\dot{X}_f$ at the end of time t_f. The path during t_0 and t_f must minimize the following quadratic functional dependent on position and rate variables, and control inputs.

$$J=\int_{t_0}^{t_f}\frac{1}{2}\left[X^TQ_1X+\dot{X}^TQ_2\dot{X}+U^TRU\right]dt \tag{2}$$

where Q_1, Q_2, and R are $(n\times n)$ symmetric matrices and R is a positive definite matrix.

The conventional way of handling this problem in the literature ([2], [5]) is to transform the n second-order differential equations (1) to $2n$ first-order differential equations by defining an n-dimensional vector $\overline{X}$, where $\overline{X}=\dot{X}$. Eq. (1) can then be written as:

$$\begin{bmatrix}\dot{X}\\ \dot{\overline{X}}\end{bmatrix}=\begin{bmatrix}0 & I\\ -M^{-1}K & -M^{-1}C\end{bmatrix}\begin{bmatrix}X\\ \overline{X}\end{bmatrix}+\begin{bmatrix}0\\ M^{-1}\end{bmatrix}U \tag{3}$$

The above matrix differential equation can be rewritten in the following compact form:

$$\dot{\mathrm{X}}=A\mathrm{X}+BU \tag{4}$$

where A is $(2n\times 2n)$ matrix, X is a $(2n\times 1)$ vector, and B is a $(2n\times n)$ matrix with entries of Eq. (3). With this redefinition of the variables, the cost function can be rewritten as

$$J=\int_{t_0}^{t_f}\left[\frac{1}{2}\left(\mathrm{X}^TQ\mathrm{X}+U^TRU\right)\right]dt \tag{5}$$

where Q is a $(2n\times 2n)$ matrix with two diagonal subblocks consisting of Q_1 and Q_2.

Using the fundamentals of variational optimization ([2], [4], [7]), the necessary conditions for extremum are obtained by defining a new functional J'

$$J'=\int_{t_0}^{t_f}\left[\frac{1}{2}\left(\mathrm{X}^TQ\mathrm{X}+U^TRU\right)+\Lambda^T\left(\dot{\mathrm{X}}-A\mathrm{X}-BU\right)\right]dt \tag{6}$$

where Λ is a $(2n\times 1)$ vector of Lagrange multipliers associated with the $2n$ differential Eqs. (4). The vector Λ^T has the form $\left(\lambda^T,\overline{\lambda}^T\right)$, where λ is associated with the first n differential equations of (4) and $\overline{\lambda}$ with the last n.

It is easy to prove that the extremal path satisfies Eq. (4) along with the following conditions:

$$\dot{\Lambda}=-Q\mathrm{X}-A^T\Lambda \tag{7}$$

$$RU+B^T\Lambda=0 \tag{8}$$

On solving for $U=-R^{-1}B^T\Lambda$ from (8) and substituting in (4), the necessary conditions become

$$\dot{\mathrm{X}}=A\mathrm{X}-BR^{-1}B^T\Lambda \tag{9}$$

$$\dot{\Lambda}=-Q\mathrm{X}-A^T\Lambda \tag{10}$$

with boundary conditions specified on X at t_0 and t_f.

These $4n$ first-order differential equations are traditionally solved either by matrix exponential methods or shooting methods [2]. Since neither the initial nor the final conditions on the costate variables are specified in the problem, in both of these methods, the objectives is to first compute these values consistent with the boundary conditions on the state variables.

2. 4^{th}-Order TPBVP Formulation

It will be shown in this section that an alternative form of the optimality equations can be written in which the state variables appear in their fourth derivative and the costate variables are eliminated. This can be achieved by adopting either of the two procedures: (i) eliminate the costate variables from Eqs. (9) and (10), (ii) rework the variational form without Lagrange multipliers by substituting U from Eq. (1) in (2). It will be shown in this section that both procedures lead to the same result. However, the procedure with Lagrange multipliers must be avoided since it may be hard, or even impossible, to eliminate these multipliers, specially, in nonlinear problems.

2.1 Elimination of Lagrange Multipliers

On eliminating Lagrange multiplier by finding Λ and $\dot{\Lambda}$ from (9) and substituting into (10), it can be shown in a few steps that Eq. (9) and (10) can be rewritten as

$$S_4\ddddot{X}+S_3\dddot{X}+S_2\ddot{X}+S_1\dot{X}+S_0X=0 \tag{11}$$

where $S_i, i=0,1,\ldots,4$ are $(n\times n)$ matrices defined as

$$S_4 = MRM$$
$$S_3 = MRC - CRM$$
$$S_2 = MRK + KRM - CRC - Q_2$$
$$S_1 = KRC - CRK$$
$$S_0 = KRK + Q_1 \tag{12}$$

2.2 Alternative Variational Form

On substituting U from Eq.(1) into Eq. (2), the cost function takes the following variational form with prescribed end conditions on x_i and $\dot{x}_i$ at t_0 and t_f:

$$J = \int_{t_0}^{t_f} \frac{1}{2}\Big[X^T Q_1 X + \dot{X}^T Q_2 \dot{X} + (M\ddot{X} + C\dot{X} + KX)^T R(M\ddot{X} + C\dot{X} + KX)\Big] dt \tag{13}$$

which can be further simplified to

$$J = \int_{t_0}^{t_f} \frac{1}{2}\Big[\ddot{X}^T MRM\ddot{X} + 2\ddot{X}^T MRC\dot{X} + 2\ddot{X}^T MRKX + 2\dot{X}^T CRKX + \dot{X}^T(CRC + Q_2)\dot{X} + X^T(KRK + Q_1)X\Big] dt \tag{14}$$

Mathematically, the form of J is

$$J = \int_{t_0}^{t_f} F(x_1,\dots,x_n,\dot{x}_1,\dots,\dot{x}_n,\ddot{x}_1,\dots,\ddot{x}_n)\, dt \tag{15}$$

and the variation of such a functional is

$$\delta J = \int_{t_0}^{t_f}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}_i}\right) h_i\, dt + \sum_{i=1}^{n}\left(\frac{\partial F}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial F}{\partial \ddot{x}_i}\right) h_i\, dt\Bigg|_{t_0}^{t_f} + \sum_{i=1}^{n}\frac{\partial F}{\partial \ddot{x}_i}\dot{h}_i\, dt\Bigg|_{t_0}^{t_f} \tag{16}$$

where $h_i(t)$ are the variations $x_i(t)$. Since h_i and $\dot{h}_i$ must vanish at the two end point, the necessary conditions for optimization become

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}_i} = 0, \quad i = 1,\dots,n \tag{17}$$

On applying the necessary condition (17) on (14), it can be shown that this yield the same optimality condition listed in (11) and (12).

3. Weighted Residual Methods

Different forms of weighted residual methods have been used to solve boundary value problems. A summary of these methods is available [1]. These methods can be classified as: (a) those that satisfy the differential equations approximately over the domain but satisfy the boundary conditions exactly such as Galerkin's method, method of moments, collocation method, and method of sub-regions, (b) weal formulations which satisfy the differential equations only partially, and (c) boundary element methods which satisfy the differential equations exactly over the domain but boundary conditions only approximately such as Trefftz method.

Due to the nature of our optimization problem with fixed end point constraints at t_0 and t_f, only the methods classified in category (a) were considered suitable. The underlying fundamental behind this method can be summarized using the following simple example. Consider the problem

$$\eta(x) - p = 0 \tag{18}$$

where $\eta(\ )$ is a differential operator, x is a function of time, and p is a constant. The solution $x(t)$ must also satisfy stated boundary conditions at the initial and final time. In this method, $x(t)$ is approximated as

$$x(t) = \sum_{i=1}^{n}\alpha_i\,\phi_i(t) \tag{19}$$

where $\alpha_i(t)$ are undetermined parameters and $\phi_i(t)$ are linearly independent mode functions selected from a complete set of functions. These functions are usually chosen to satisfy admissibility conditions relating to the boundary conditions. On substituting (19) in (18), the following error function results

$$\varepsilon = \eta(x) - p \tag{20}$$

This error function $\varepsilon(t)$ is forced to be zero, in the average sense, by setting weighted integrals of the residual equal to zero, i.e.,

$$\int_{t_0}^{t_f}\varepsilon\,\psi_i\, dt = 0 \tag{21}$$

where $\psi_i(t)$ are the weighting functions.

The category (a) methods differ primarily in their selection of weighting functions. For example, the method of moments uses weighting function as $t^i,\ i = 1,\dots,n$. Galerkin's method uses weighting functions the same as mode functions. In this paper, Galerkin's method was selected to obtain the approximate solution of the problem because of its generality and ubiquitous use in solving problems of mechanics. The mode functions in this problem are chosen as polynomials due to their simplicity of analytical integration.

4. Galerkin's Solution

The approximate solution of the fourth-order differential Eq. (11) must be obtained subjected to the following boundary conditions $X(t_0) = X_0$, $\dot{X}(t_0) = \dot{X}_0$, $X(t_f) = X_f$, and $\dot{X}(t_f) = \dot{X}_f$. In order to ensure admissibility of the trial functions, the approximate solution must have the following form [3]:

$$X(t) = \Phi_0(t) + \sum_{i=1}^{m} L_i \phi_i(t) \tag{22}$$

where $\Phi_0(t)$ is an n-dimensional vector of mode functions that satisfies the boundary conditions of the vector X at time t_0 and t_f. $\phi_i(t)$ are mode functions, which vanish at the two end points and also have zero derivatives at the end points. As a result, $X(t)$ always satisfies the boundary conditions of the problem. $L_1,\dots,L_m$ are n-dimensional constant vectors that are determined by minimizing the residual error.

On substituting (22) in Eq. (11), the following error vector results:

$$\begin{aligned}
\varepsilon(t) &= s_4 \ddddot{\Phi}_0 + s_3\dddot{\Phi}_0 + s_2\ddot{\Phi}_0 + s_1\dot{\Phi}_0 + s_0\Phi_0 \\
&+ L_1\left(s_4 \ddddot{\Phi}_1 + s_3\dddot{\Phi}_1 + s_2\ddot{\Phi}_1 + s_1\dot{\Phi}_1 + s_0\Phi_1 \right) \\
&+ L_2\left(s_4 \ddddot{\Phi}_2 + s_3\dddot{\Phi}_2 + s_2\ddot{\Phi}_2 + s_1\dot{\Phi}_2 + s_0\Phi_2 \right) \\
&+ \dots \\
&+ L_m\left(s_4 \ddddot{\Phi}_m + s_3\dddot{\Phi}_m + s_2\ddot{\Phi}_m + s_1\dot{\Phi}_m + s_0\Phi_m \right)
\end{aligned} \tag{23}$$

In accordance with Galerkin's procedure, the error function must be chosen to be orthogonal to the mode functions

$$\int_{t_0}^{t_f} \varepsilon(t)\phi_i(t)\, dt = 0, \quad i = 1,\dots,m \tag{24}$$

This leads to mn scalar equations which can be used to solve for the mn elements of the vector $L_1,\dots,L_m$. The equation (25) can be written in a matrix form:

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ T_{21} & T_{22} & \cdots & T_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mm} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix} = \begin{bmatrix} -R_1 \\ -R_2 \\ \vdots \\ -R_m \end{bmatrix} \tag{25}$$

where T_{pl} is a $(n \times n)$ matrix subblock, and R_p is a $(n \times 1)$ vector defined below:

$$\begin{aligned}
T_{pl} &= \int_{t_0}^{t_f} \left(s_4 \ddddot{\Phi}_l + s_3\dddot{\Phi}_l + s_2\ddot{\Phi}_l + s_1\dot{\Phi}_l + s_0\Phi_l \right)\phi_p \, dt \\
R_p &= \int_{t_0}^{t_f} \left(s_4 \ddddot{\Phi}_0 + s_3\dddot{\Phi}_0 + s_2\ddot{\Phi}_0 + s_1\dot{\Phi}_0 + s_0\Phi_0 \right)\phi_p \, dt
\end{aligned} \tag{26}$$

The above equation can be inverted to solve for the vectors $L_1,\dots,L_m$.

4.1 Mode Functions: A Particular Choice

It is quite evident that any set of $\Phi_0(t)$ and $\phi_i(t)$ that satisfies the boundary conditions is a valid set of mode functions. In this paper, $\Phi_0(t)$ is chosen as the following cubic function of time

$$\begin{aligned}
\Phi_0 &= x_0 + \dot{x}_0 t + \left[\frac{3}{t_f^2}\left(x_f - x\right)_0 - \frac{2}{t_f}\dot{x}_0 - \frac{1}{t_f}\dot{x}_f \right] t^2 \\
&+ \left[\frac{2}{t_f^3}\left(x_f - x_0\right) + \frac{1}{t_f^2}\left(\dot{x}_f - \dot{x}_0\right) \right] t^3
\end{aligned} \tag{27}$$

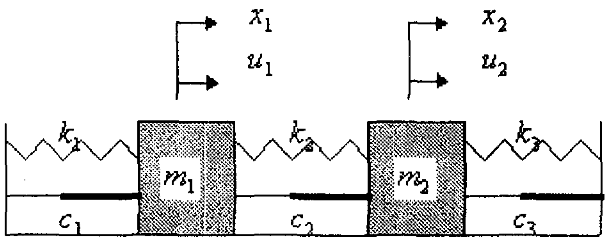

Figure 1: A two degrees-of-freedom spring-mass-damper system.

It can be easily verified that $\Phi_0(t_0) = X_0$, $\Phi_0(t_f) = X_f$, $\dot{\Phi}_0(t_0) = \dot{X}_0$, and $\dot{\Phi}_0(t_f) = \dot{X}_f$. The mode functions $\phi_i(t)$ are selected as

$$\phi_i(t) = t^2\left(t - t_f\right)^{i+1} \quad , i = 1, \ldots, m \qquad (28)$$

These mode functions possess the properties $\phi_i(t_0) = \phi_i(t_f) = \dot{\phi}_i(t_0) = \dot{\phi}_i(t_f) = 0$.

With these mode functions, the matrix T_{pl} and the right hand side vector R_p can be analytically computed, respectively.

5. Example

A two degree-o-freedom spring-mass-damper system is used as an example. The system is sketched in Figure 1. The matrices M, C, and K for this system are:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \qquad (29)$$

The states are commanded to move from $X_0 = (10,20)^T$, $\dot{X}_0 = (10,20)^T$ to the equilibrium position with zero final velocity, i.e., $X_f = \dot{X}_f = 0$. The parameters of (29) in MKS units are: $m_1 = m_2 = 1.0$, $c_1 = c_3 = 1.0$, $c_2 = 2.0$, $k_1 = k_2 = k_3 = 3.0$. The matrices in the cost function are: $R = I_2, Q_1 = Q_2 = 0$.

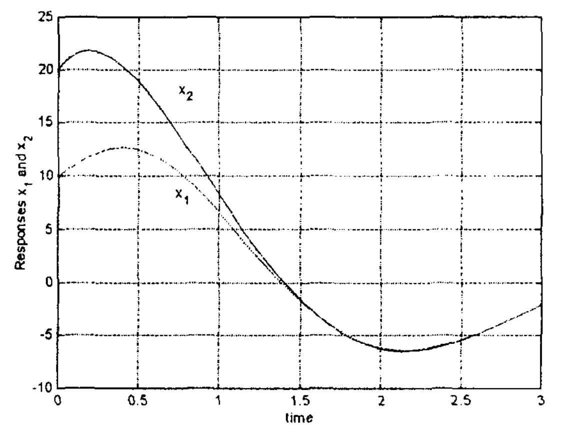

Figure 2: The optimal response curves: Comparison of matrix exponential solution and Galerkin solution for $t_f = 3$ seconds.

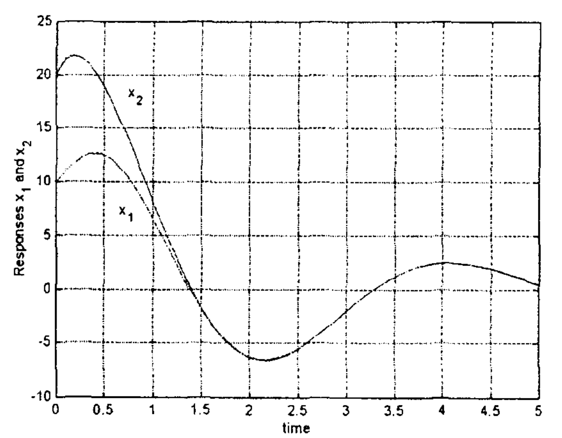

Figure 3: The optimal response curves: Comparison of matrix exponential solution and Galerkin solution for $t_f = 5$ seconds.

Figure 2 compares the optimal response curves of the state variables x_1 and x_2 obtained using matrix exponential method and Galerkin's method for $t_f = 3$ seconds. From these response curves, it is quite evident that with only three modes, Galerkin's solution becomes very close and overlap with matrix exponential solution.

Figure 3 again compares the response curves for identical parameters but $t_f = 5$ seconds. The number of modes required for a reasonably accurate solution is higher in this case (5 modes) due to more zero crossings over $t_f = 5$ seconds of the response. It is clear that Galerkin's solution becomes very close and overlap with matrix exponential solution.

6. Conclusion

An alternative fourth-order two-point boundary value formulation has been presented for the solution of dynamic optimization of linear springs, mass, damper systems. The fourth-order formulation requires solution of n fourth-order equations in state variables for an n degrees-of-freedom system while the conventional approaches in the literature solve the same problem using $4n$ first-order differential equations in the state and the costate variables. The solution approach using the conventional way becomes difficult and computation intensive because the initial or final conditions on the costate variable are not known and must be computed iteratively. Due to the absence of costate variables in the fourth-order formulation, Galerkin's method becomes highly attractive to solve the problem. This paper outlines the Galerkin's solution approach and shows how quickly the solutions can be obtained by only using a few modes. The author believe that the fourth-order formulation combined with Galerkin's approach will offer computational benefits for optimization of nonlinear dynamic systems to obtain the exact solution as well as to quickly arrive at the approximate solution which can be used as initial guess to algorithms based on shooting methods. The fourth-order TPBVP formulation is extremely attractive for real-time control and motion planning due to its simple computational procedure.

References

[1] Brebbia, C.A., *The Boundary Element Method for Engineers*, Pentech Press, 1978.

[2] Bryson, A.E. and Ho, Y.C., *Applied Optimal Control*, Hemisphere Publishing Company, 1975.

[3] Fletcher, C.A.J., *Computational Galerkin Methods*, Springer Verlag, 1984.

[4] Gelfand, I.S. and Fomin, S.V., *Calculus of Variations*, Prentice-Hall Book Company, 1963.

[5] Kirk, D.E., *Optimal Control Theory: An Introduction*, Prentice-Hall Electrical Engineering Series, 1970.

[6] Meirovitch, L., *Analytical Methods in Variations*, Macmillan Book Company, 1967.

[7] Pinch, E.R., *Optimal Control and the Calculus of Variations*, Oxford University Press, 1993.