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Determining Boundary Heat Flux from Boundary Temperature Measurements

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Abstract

Temperature measurements at boundary are used to determine boundary heat flux applied to a multidimensional body. The boundary element method is used as the method of solution because it provides direct relationship between boundary heat flux and boundary temperature. A sample problem is shown to demonstrate the effectiveness of this technique.

1. Introduction

Conventional measurement of boundary heat flux is usually performed directly by using a sophisticated device such as Sandwich-type heat flux gauge, circular foil heat flux gauge, or calorimetric gauge [1]. This not only makes the measurement expensive but also introduces large errors. On the other hand, surface temperature measurement is afforded by several well-established techniques. Some of them do not require expensive devices. The possibility of determining heat flux from temperature measurements has long been recognized [2]. However, due to the fact, in addition to temperature measurements, the solution to the relevant heat conduction problem is needed, this technique of measuring heat flux has not been as widely paid attention to as it should be. The problem of determining boundary heat flux of a multidimensional body from interior temperature measurements is known as the inverse heat conduction problem. A few numerical techniques have been proposed for solving such a problem [3-5]. It is well known that the solution to the inverse heat conduction problem with interior temperature measurements as input data is subjected to instability. However, if boundary temperature measurements are used instead, the solution will be stable. Moreover, using boundary temperature measurements as input data in estimating boundary heat flux is advantageous in that it is easier to implement in practice and that the numerical solution can be made efficient by employing the boundary element method, which relate boundary temperatures explicitly to boundary heat flux components. This paper will present a formulation of the boundary element method for solving the inverse heat conduction problem. It will be shown that this method is efficient and capable of giving an accurate solution.

2. Statement of the Problem

Consider a solid object with part of its boundary Γ_1 subjected to known heat flux and the remaining part of the boundary Γ_2 subjected to unknown heat flux. Suppose that the object has constant thermophysical properties, making the problem a linear one. Without the loss of generality, we can take the value of the thermal diffusivity to be unity and the initial condition to be uniformly zero. The heat conduction process can then be described by the following equations.

$$\frac{\partial T(\vec{r},t)}{\partial t} = \nabla^2 T(\vec{r},t) \quad (1)$$

$$T(\vec{r},0) = 0 \quad (2)$$

$$\vec{n}\vec{\nabla}T(\vec{r},t)\big|_{\Gamma_1} = g(\vec{r},t)\big|_{\Gamma_1} \quad (3)$$

where $\vec{n}$ is the outward pointing unit vector normal to boundary and g is the known boundary heat flux. In order to render the problem solvable, the temperature measurement data must be specified.

$$T(\vec{r}_i, j\Delta t) = T_i^{(j)} \quad (4)$$

where $\vec{r}_i$ is a sensor position vector, and Δt is the measurement time step. The temperature sensors are located on the boundary.

3. Boundary Element Method

The boundary element formulation for a transient linear heat conduction problem is given by [6]

$$aT(\vec{\xi},t) = \int_\Gamma\int_0^t q(\vec{r},\tau)G(\vec{r}-\vec{\xi};t-\tau)d\tau d\vec{r} - \int_\Gamma\int_0^t T(\vec{r},\tau)\vec{n}\vec{\nabla}G(\vec{r}-\vec{\xi};t-\tau)d\tau d\vec{r} \quad (5)$$

where a depends on the location of $\vec{\xi}$, and the fundamental solution

$$G(\vec{r}-\vec{\xi};t-\tau) = \frac{e^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{[4\pi(t-\tau)]^{m/2}} \quad (6)$$

and m is the dimension of the problem. Divide the boundary Γ into M_e boundary elements and time t into N equal time intervals. Eq. (5) becomes

$$aT(\vec{\xi},t) = \sum_{i=1}^{M_e}\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t} {}_iq(\vec{r},\tau)G(\vec{r}-\vec{\xi};N\Delta t-\tau)d\tau\right]d\vec{r}$$

$$-\sum_{i=1}^{M_e}\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t} {}_iT(\vec{r},\tau)\vec{n}\vec{\nabla}G(\vec{r}-\vec{\xi};N\Delta t-\tau)d\tau\right]d\vec{r} \quad (7)$$

where front subscript denotes element index. Now, let's approximate ${}_iq$ and ${}_iT$ by piecewise linear functions in time.

$${}_iq(\vec{r},\tau) = \frac{1}{\Delta t}\left[{}_iq^{(j)}(\vec{r}) - {}_iq^{(j-1)}(\vec{r})\right](\tau - N\Delta t) +$$

$${}_iq^{(j)}(\vec{r})(N-j+1) - {}_iq^{(j-1)}(\vec{r})(N-j) \quad (8)$$

$${}_iT(\vec{r},\tau) = \frac{1}{\Delta t}\left[{}_iT^{(j)}(\vec{r}) - {}_iT^{(j-1)}(\vec{r})\right](\tau - N\Delta t) +$$

$${}_iT^{(j)}(\vec{r})(N-j+1) - {}_iT^{(j-1)}(\vec{r})(N-j) \quad (9)$$

where superscript denotes time index. Next, approximate ${}_iq^{(j)}$ and ${}_iT^{(j)}$ over element i, making use of interpolating function Φ_k, as follows.

$${}_iq^{(j)}(\vec{r}) = \sum_{k=1}^{L} {}_{i,k}q^{(j)}\Phi_k(\vec{r}) \quad (10)$$

$${}_iT^{(j)}(\vec{r}) = \sum_{k=1}^{L} {}_{i,k}T^{(j)}\Phi_k(\vec{r}) \quad (11)$$

where k is local node index, and L is the number of nodes in an element. Substituting Eqs. (8)-(11) into Eq. (7) yields

$$aT(\vec{\xi},t) = \sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + \right.\right.\right.$$

$$\left.\left.\left.(N-j+1)\right)G d\tau\right]\Phi_k(\vec{r})d\vec{r}\right\}\left({}_{i,k}q^{(j)}\right)$$

$$-\sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + \right.\right.\right.$$

$$\left.\left.\left.(N-j)\right)G d\tau\right]\Phi_k(\vec{r})d\vec{r}\right\}\left({}_{i,k}q^{(j-1)}\right)$$

$$-\sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + \right.\right.\right.$$

$$\left.\left.\left.(N-j+1)\right)\vec{n}\vec{\nabla}G d\tau\right]\Phi_k(\vec{r})d\vec{r}\right\}\left({}_{i,k}T^{(j)}\right)$$

$$+\sum_{i=1}^{M_e}\sum_{k=1}^{L}\left\{\int_{\Gamma_i}\left[\sum_{j=1}^{N}\int_{(j-1)\Delta t}^{j\Delta t}\left(\frac{(\tau-N\Delta t)}{\Delta t} + \right.\right.\right.$$

$$\left.\left.\left.(N-j)\right)\vec{n}\vec{\nabla}G d\tau\right]\Phi_k(\vec{r})d\vec{r}\right\}\left({}_{i,k}T^{(j-1)}\right) \quad (12)$$

If Eq. (12) is evaluated at a point $\vec{\xi}_k$ on the boundary or inside the object, the resulting equation after the assembly process can be written as

$$a_kT_k^{(N)} = \sum_{j=1}^{N}\sum_{i=1}^{M_n+M_c}\phi(\vec{\xi}_k,\vec{r}_i,(N-j)\Delta t)T_i^{(j)}$$

$$+\sum_{j=1}^{N}\sum_{i=1}^{M_n}\psi(\vec{\xi}_k,\vec{r}_i,(N-j)\Delta t)q_i^{(j)} \quad (13)$$

where back subscript denotes global node index, M_n is the number of boundary nodes, and M_c is the number of additional heat flux components at corner or edge nodes. Note that coefficient a_k becomes unity if $\vec{\xi}_k$ is inside the object. For two-dimensional problems, each corner node can have two heat flux components; therefore, M_c is equal to the number of corners. For three-dimensional problems, each edge node can have two heat flux components, and each corner node can have three heat flux components. Functions ϕ and ψ are obtained from the evaluation of integrals shown in Eq. (12). The evaluation of time integrals can be done exactly as shown in the Appendix, whereas the evaluation of boundary integrals should be performed using the Gaussian quadrature.

Equation (13) is now written for M_n boundary node points yielding M_n equations, which may be expressed as

$$A\vec{T}^{(N)} = \sum_{j=1}^{N}P^{(N-j)}\vec{T}^{(j)} + \sum_{j=1}^{N}R^{(N-j)}\vec{q}^{(j)} + \sum_{j=1}^{N}S^{(N-j)}\vec{g}^{(j)} \quad (14)$$

where A is diagonal matrix of coefficients a; $\vec{T}$ is the vector of temperatures on the boundary; $\vec{q}$ is the vector of boundary heat flux components that are to be determined; $\vec{g}$ is the vector of specified boundary heat flux components; and P, R, and S are coefficient matrices consisted of ψ and ϕ functions.

Let $\vec{T}_0$ be boundary temperature responses when $\vec{q}$ vanishes.

$$A\vec{T}_0^{(N)} = \sum_{j=1}^{N}P^{(N-j)}\vec{T}_0^{(j)} + \sum_{j=1}^{N}S^{(N-j)}\vec{g}^{(j)} \quad (15)$$

If $\vec{g}$ is known as a function of time, $\vec{T}_0$ can be found by a time-stepping procedure. Note that $\vec{T}_0 \neq 0$ only if $\vec{g} \neq 0$. Subtracting Eq. (15) from (14) results in

$$A\left[\vec{T}^{(N)} - \vec{T}_0^{(N)}\right] = \sum_{j=1}^{N} P^{(N-j)}\left[\vec{T}^{(j)} - \vec{T}_0^{(j)}\right] + \sum_{j=1}^{N} R^{(N-j)}\vec{q}^{(j)} \quad (16)$$

Applying a time-stepping procedure to Eq. (16) yields the following relation between boundary temperatures and the unknown boundary heat flux:

$$\vec{T}^{(k)} - \vec{T}_0^{(k)} = \sum_{j=1}^{k} X^{(k-j)}\vec{q}^{(j)} \quad (17)$$

In order to make the computation of heat flux straightforward, heat flux is expressed in terms of boundary temperatures as

$$\vec{q}^{(k)} = \sum_{j=1}^{k} Y^{(k-j)}\left(\vec{T}^{(j)} - \vec{T}_0^{(j)}\right) \quad (18)$$

where
$$Y^{(0)} = \left(X^{(0)}\right)^{-1}$$

$$Y^{(l)} = -\left(X^{(0)}\right)^{-1}\sum_{j=0}^{l-1} X^{(l-j)}Y^{(j)}$$
$$(1 \le l \le k-1) \quad (19)$$

3. Results and Discussion

The sample problem to be considered is illustrated in Fig. 1. A square object, which is insulated on three sides, is subjected to unknown heat flux on the remaining side AB. Suppose that temperature measurements are available on that side. The unknown heat flux will be determined using the algorithm described earlier.

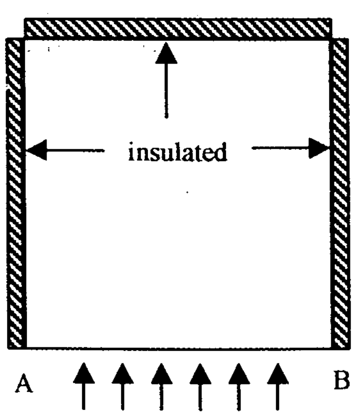

Fig. 1 Sample problem

For computational purpose, the boundary is divided into 40 elements. Since only surface AB is non-insulated, only the heat flux components on 11 nodes along AB are to be estimated from measurement data, which are obtained from 11 temperature sensors placed at the nodes. This problem is simplified by the fact that $\vec{g} = 0$. Hence, $\vec{T}_0$ vanishes. The time step used is 0.02, and the calculation is performed from time 0 to 1.2.

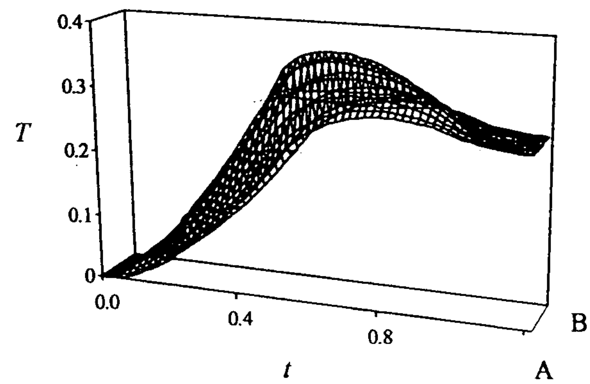

Fig. 2 Temperature measurements at surface AB

Figure 2 shows the temperature measurements at surface AB. This temperature distribution is obtained from the boundary element solution of the same problem with known heat flux, as shown in Fig. 3. If these temperature measurements are used as input data for the estimation of unknown heat flux, it is found that the determined heat flux is identical with the heat flux shown in Fig. 3. Hence, this algorithm is capable of providing an accurate estimation of unknown heat flux.

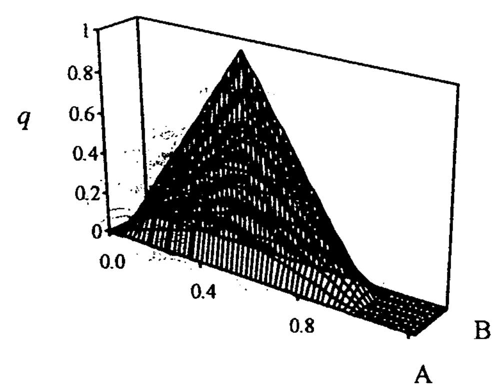

Fig. 3 Heat flux that yields temperature measurements in Fig. 2

The temperature measurements in Fig. 2 are errorless measurements, which are unlikely in reality. Usually, each measurement contains some statistical error. As a result, the estimated heat flux will contain statistical error too. Let's assume that the error in temperature measurement is normally distributed with standard deviation σ, and that errors of any two measurements are uncorrelated. The linear relation between estimated heat flux and temperature measurements in Eq. (18) implies that the ratio of the variance in estimated heat flux to the variance in temperature measurements is

$$\frac{\mathrm{Var}\left(q_i^{(k)}\right)}{\sigma^2} = \sum_{j=1}^{M_s}\sum_{l=1}^{k}\left(Y_{ij}^{(k-l)}\right)^2 \tag{20}$$

where M_s is the number of heat flux components to be determined.

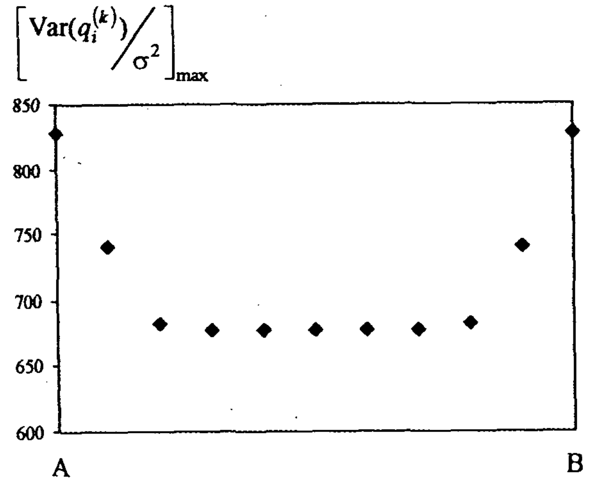

Fig. 4 Maximum variance in estimated heat flux along the surface AB

It can be seen that the variance in estimated heat flux increases monotonically with time. Hence, maximum variance is reached at maximum k. In other words, the variance of estimated heat flux component at earlier time is smaller than the variance of estimated heat flux component at later time. In addition, it should be noted that there is variation in maximum variance along the surface, as shown in Fig. 4.

4. Conclusion

A method for estimating boundary heat flux of a multidimensional body from boundary temperature measurements is presented. This method requires the numerical solution to the inverse heat conduction problem, which can be efficiently obtained with the boundary element method. The estimated heat flux is expressed as a linear function of boundary temperatures. Hence, statistical error in the estimate is readily determined if errors in measurements are known. The efficiency and simplicity of this method should make the determination of boundary heat flux from boundary temperature measurements a viable alternative to conventional methods of determining boundary heat flux.

5. Acknowledgment

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Appendix

Analytical evaluation of time integrals in Eq. (12) will now be given separately for two-dimensional and three-dimensional problems.

Two-dimensional problem

From Eq. (6),

$$G_{2D}\left(\vec r-\vec\xi;t-\tau\right) = \frac{e^{-\left(\vec r-\vec\xi\right)^2/4(t-\tau)}}{4\pi(t-\tau)} \tag{21}$$

$$\vec\nabla G_{2D}\left(\vec r-\vec\xi;t-\tau\right) = -\frac{e^{-\left(\vec r-\vec\xi\right)^2/4(t-\tau)}}{8\pi(t-\tau)^2}\left|\vec r-\vec\xi\right|\vec\nabla\left|\vec r-\vec\xi\right| \tag{22}$$

$$\int_{(j-1)\Delta t}^{j\Delta t} G_{2D}\,d\tau = \frac{1}{4\pi}\left\{E_1\left(\frac{\left(\vec r-\vec\xi\right)^2}{4(N-j+1)\Delta t}\right) - E_1\left(\frac{\left(\vec r-\vec\xi\right)^2}{4(N-j)\Delta t}\right)\right\} \tag{23}$$

$$\begin{aligned}\int_{(j-1)\Delta t}^{j\Delta t}(N\Delta t-\tau)G_{2D}\,d\tau = &-\frac{(N-j)\Delta t}{4\pi}e^{-\left(\vec r-\vec\xi\right)^2/4(N-j)\Delta t} \\ &+\frac{(N-j+1)\Delta t}{4\pi}e^{-\left(\vec r-\vec\xi\right)^2/4(N-j+1)\Delta t} \\ &-\frac{\left(\vec r-\vec\xi\right)^2}{16\pi}\left\{E_1\left(\frac{\left(\vec r-\vec\xi\right)^2}{4(N-j+1)\Delta t}\right) - E_1\left(\frac{\left(\vec r-\vec\xi\right)^2}{4(N-j)\Delta t}\right)\right\}\end{aligned} \tag{24}$$

$$\int_{(j-1)\Delta t}^{j\Delta t}\vec\nabla G_{2D}\,d\tau = -\frac{\left|\vec r-\vec\xi\right|\vec\nabla\left|\vec r-\vec\xi\right|}{2\pi\left(\vec r-\vec\xi\right)^2}\left\{e^{-\left(\vec r-\vec\xi\right)^2/4(N-j+1)\Delta t} - e^{-\left(\vec r-\vec\xi\right)^2/4(N-j)\Delta t}\right\} \tag{25}$$

$$\int_{(j-1)\Delta t}^{j\Delta t}(N\Delta t-\tau)\vec{\nabla}G_{2D}\mathrm{d}\tau \quad = $$

$$\frac{|\vec{r}-\vec{\xi}|\vec{\nabla}|\vec{r}-\vec{\xi}|}{8\pi}\left\{\mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j)\Delta t}\right) - \mathrm{E}_1\left(\frac{(\vec{r}-\vec{\xi})^2}{4(N-j+1)\Delta t}\right)\right\} \tag{26}$$

where E_1 is exponential integral, defined as

$$\mathrm{E}_1(x) \quad = \quad \int_x^{\infty}\frac{\mathrm{e}^{-y}}{y}\mathrm{d}y$$

Three-dimensional problem

From Eq. (6),

$$G_{3D}(\vec{r}-\vec{\xi};t-\tau) \quad = \quad \frac{\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{[4\pi(t-\tau)]^{3/2}} \tag{27}$$

$$\vec{\nabla}G_{3D}(\vec{r}-\vec{\xi};t-\tau) \quad = \quad -\frac{\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(t-\tau)}}{16\pi^{3/2}(t-\tau)^{5/2}}|\vec{r}-\vec{\xi}|\vec{\nabla}|\vec{r}-\vec{\xi}| \tag{28}$$

$$\int_{(j-1)\Delta t}^{j\Delta t}G_{3D}\mathrm{d}\tau \quad = \quad \frac{1}{4\pi|\vec{r}-\vec{\xi}|}\left\{\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\right) - \mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\right)\right\} \tag{29}$$

$$\int_{(j-1)\Delta t}^{j\Delta t}(N\Delta t-\tau)G_{3D}\mathrm{d}\tau$$

$$= \quad \frac{\sqrt{(N-j+1)\Delta t}}{4\pi^{3/2}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j+1)\Delta t}$$

$$\frac{\sqrt{(N-j)\Delta t}}{4\pi^{3/2}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j)\Delta t} \; -$$

$$\frac{|\vec{r}-\vec{\xi}|}{8\pi}\left\{\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\right) - \mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\right)\right\} \tag{30}$$

$$\int_{(j-1)\Delta t}^{j\Delta t}\vec{\nabla}G_{3D}\mathrm{d}\tau \quad =$$

$$\frac{\vec{\nabla}|\vec{r}-\vec{\xi}|}{2\pi^{3/2}(\vec{r}-\vec{\xi})^2}\left\{\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j)\Delta t} - \frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\mathrm{e}^{-(\vec{r}-\vec{\xi})^2/4(N-j+1)\Delta t} -\right.$$

$$\left.\frac{2}{\sqrt{\pi}}\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\right) + \frac{2}{\sqrt{\pi}}\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\right)\right\} \tag{31}$$

$$\int_{(j-1)\Delta t}^{j\Delta t}(N\Delta t-\tau)\vec{\nabla}G_{3D}\mathrm{d}\tau \quad =$$

$$-\frac{\vec{\nabla}|\vec{r}-\vec{\xi}|}{8\pi}\left\{\mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j+1)\Delta t}}\right) - \mathrm{erfc}\left(\frac{|\vec{r}-\vec{\xi}|}{2\sqrt{(N-j)\Delta t}}\right)\right\} \tag{32}$$

where erfc is complementary error function, defined as

$$\mathrm{erfc}(x) \quad = \quad \frac{2}{\sqrt{\pi}}\int_x^{\infty}\mathrm{e}^{-y^2}\mathrm{d}y$$