# วิธีการกึ่งวิเคราะห์ปัญหาแหล่งน้ำมันที่มีอาณาบริเวณเป็นวงกลมโดยมีเงื่อนไขขอบแบบไม่มีการไหล และ มีความดันคงที่: กรณีเฉพาะหลุมที่มีรอยแตก

# Semi-Analytical Solutions for a Bounded Circular Reservoir with No Flow and Constant Pressure Outer Boundary: Fractured Well Case

ร.ท.พิสิษฐ์ ลิขิตสุภิณ* และ ดร. โทมัส เอ บลาสซิงเกม**

Lt. Phisit Likitsupin and Thomas A. Blasingame

## บทคัดย่อ

งานวิจัยนี้เกี่ยวกับการหาสมการคำตอบโดยใช้วิธีการกึ่งวิเคราะห์ สำหรับหลุมที่มีรอยแตกซึ่งผลิตในอัตราการไหลที่คงที่จากอาณาบริเวณซึ่งเป็นวงกลม โดยมีเงื่อนไขขอบ แบบไม่มีการไหล หรือมีความดันคงที่ ประโยชน์ของการค้นพบคือทำให้มีสมการในรูปที่สะดวกต่อการคำณวนหาสถานะภาพของแหล่งน้ำมัน ซึ่งปัจจุบันสมการมีอยู่เฉพะในรูปของสมการลาปลาสที่มีเบสเซลเทอมที่ยุ่งยาก และต้องใช้การคำณวนเชิงตัวเลขมาใช้ในการหาคำตอบ

สำหรับสมการคำตอบที่อยู่ในรูปของสมการลาปลาสแม้จะมีเทอมที่ยุ่งยาก แต่ยังง่ายต่อการคำณวนโดยใช้คอมพิวเตอร์ อย่างไรก็ตามงานวิจัยนี้ช่วยให้ผู้สนใจได้เห็นเด่นชัดขึ้นในหลักและที่มาของแต่ละส่วนของสมการ และสามารถนำสมการไปใช้ในการคำณวนหาสถานะภาพของแหล่งน้ำมัน รวมถึงการนำไปใช้ในการวิเคราะห์หาคุณลักษณะของแหล่งน้ำมันจากข้อมูลที่ได้รับ

## ABSTRACT

This paper concerns the development of semi-analytical (approximate) solutions for a fractured well producing at a constant flow rate from the center of a bounded circular reservoir that has a no flow or constant pressure outer boundary. The utility of these solutions is that they provide explicit formulae for computing reservoir performance, as proposed to the Laplace transform solutions which have complicated Bessel functions and require numerical inversion.

The more rigorous (and hence, complex) Laplace transform solutions are relative easy to compute and manipulate given modern computing environments; however, this paper serves as mechanism to provide those interested with accurate and computational simple methods for

* The leoturer of Meohanioal Engineering Department, Chulaohomklao Royal Military Aoademy, Thailand

** The Assistance Professor of Petroleum Engineering Department, Texas A&M University, U.S.A.

illustrating concepts as well as for computing reservoir performance and for the analysis and interpretation of reservoir performance data.

1. INTRODUCTION

Fracturing is an effective technique to increase the productivity of damaged wells, or from wells of low flow capacity. Gringarten, *et al*[1] introduced the concept of infinite conductivity vertical fracture and generated type curves to facilitate modeling and analysis using these solutions. At present, the finite fracture conductivity concept is considered to be most appropriate. However, the infinite conductivity fracture constitutes a special case and is valid for $C_{fD} > 100$. The solutions for infinite conductivity case are solutions with $x_D = 0$. In case of a high, but not infinite, fracture conductivity, the same solution with $x_D = 0.732$ (uniform flux) can be used[1].

In case of infinite fracture conductivity, there is no flow resistance in the fracture itself. The pressure distribution in a reservoir is described by a radial flow equation as:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial p_D}{\partial r_D}\right)=\frac{\partial p_D}{\partial t_{LfD}} \quad \text{.......(1)}$$

The initial condition describes the uniformity of pressure throughout the reservoir:

$$p_D\left(r_D,t_{LfD}=0\right)=0 \quad \text{.......(2)}$$

The continuity between the two flow regions in the fracture model and the plane source of the pressure and the flux density (fluid rate per unit fracture length) may be expressed as:

$$p_{fD}\left(x_D,t_{LfD}\right)=p_D\left(x_D,t_{LfD}\right),\quad -1\le x_D\le 1,\ t_{LfD}>0 \quad \text{.......(3)}$$

and:

$$q_{fD}\left(x_D,t_{LfD}\right)=q_D\left(x_D,t_{LfD}\right),\quad -1\le x_D\le 1,\ t_{LfD}>0 \quad \text{.......(4)}$$

Okan and Ranghavan[2] consider a Green's function approach of integrating a line source along a path (fracture) in the reservoir. The Laplace transform solutions are for a fractured well with a constant pressure outer boundary as:

$$\bar{p}_{D,cpb}\left(u,r_{eD},x_D\right)=\frac{1}{2u}\int_{-1}^{1}\left[K_0\left\{\sqrt{u}\left(x_D-z\right)\right\}-\frac{K_0\left(\sqrt{u}r_{eD}\right)}{I_0\left(\sqrt{u}r_{eD}\right)}I_0\left\{\sqrt{u}\left(x_D-z\right)\right\}\right]dz \quad \text{.......(5)}$$

and with a no flow outer boundary as:

$$\bar{p}_{D,nfb}\left(u,r_{eD},x_D\right)=\frac{1}{2u}\int_{-1}^{1}K_0\left\{\sqrt{u}\left(x_D-z\right)\right\}dz+\frac{K_1\left(\sqrt{u}r_{eD}\right)}{2uI_1\left(\sqrt{u}r_{eD}\right)}\left[\int_0^{\sqrt{u}(1-x_D)}I_0(z)dz+\int_0^{\sqrt{u}(1+x_D)}I_0(z)dz\right] \quad \text{.......(6)}$$

The $\frac{1}{2u}\int_{-1}^{1}K_0\left\{\sqrt{u}\left(x_D-z\right)\right\}dz$ term from both equations is the solution for a fractured well in an infinite acting reservoir. The inversion of this term is given by Gingarten[3], *et al* as:

$$p_{D,\inf}\left(x_D,0,t_{LfD}\right)=\frac{\sqrt{\pi t_{LfD}}}{2}\left[\mathrm{erf}\left(\frac{1-x_D}{2\sqrt{t_{LfD}}}\right)+\mathrm{erf}\left(\frac{1+x_D}{2\sqrt{t_{LfD}}}\right)\right]+\frac{1-x_D}{4}\mathbf{E}_1\left(\frac{(1-x_D)^2}{4t_{LfD}}\right)+\frac{1+x_D}{4}\mathbf{E}_1\left(\frac{(1+x_D)^2}{4t_{LfD}}\right) \quad (7)$$

and the derivative is given as:

$$p'_{D,\inf}\left(x_D,t_{LfD}\right)=\frac{\sqrt{\pi t_{LfD}}}{4}\left[\mathrm{erf}\left(\frac{1-x_D}{2\sqrt{t_{LfD}}}\right)+\mathrm{erf}\left(\frac{1+x_D}{2\sqrt{t_{LfD}}}\right)\right] \quad (8)$$

From Eqs. (5) and (6), the bounded dominated parts are in Laplace domain, which have to be evaluated numerically. The approach to obtain invertable forms of the solutions will rely on polynomial expansions of the modified Bessel functions of the first kind $\{I_0(z)$ and $I_1(z)\}$, several forms of a solution can be developed by extending or truncating a particular set of terms. For any developed relation the following results should be verified:

- The approximate real space solution is accurate compared to the numerical inversion of the Laplace transform solution.
- The derivative of the approximate real space solution is accurate compared to the derivative function obtained from numerical inversion of the Laplace transform solution.

## 2. LAPLACE TRANSFORM SOLUTIONS AND REAL SPACE APPROXIMATE SOLUTIONS

- Constant Pressure Outer Boundary Case

The outer boundary condition for constant pressure outer boundary is:

$$p_D\left(x_D,t_{LfD},r_{eD}\right)=0 \quad (9)$$

The Laplace domain solution for an unfractured well in an infinite acting reservoir is given by Okan and Ranghavan[2] as:

$$\bar{p}_{D,cpb}(u,r_{eD},x_D)=\frac{1}{2u}\int_{-1}^{1}\left[K_0\left\{\sqrt{u}(x_D-z)\right\}-\frac{K_0\left(\sqrt{u}r_{eD}\right)}{I_0\left(\sqrt{u}r_{eD}\right)}I_0\left\{\sqrt{u}(x_D-z)\right\}\right]dz \quad (10)$$

By inspection of Eq. (10), the $\frac{1}{2u}\int_{-1}^{1}K_0\left\{\sqrt{u}(x_D-z)\right\}dz$ term is the constant rate solution for a fractured well in an infinite-acting homogeneous reservoir. The solution that need to be found is the finite-acting part. Let $y=-\sqrt{(x_D-z)^2}$, the upper and lower limits of the integral will be rewritten as:

$$\bar{p}_{D,cpb}(u,r_{eD},x_D)=\bar{p}_{D,\inf}(u,x_D)-\frac{K_0\left(\sqrt{u}r_{eD}\right)}{2u\sqrt{u}I_0\left(\sqrt{u}r_{eD}\right)}\left[\int_0^{\sqrt{u}(1-x_D)}I_0(z)dz+\int_0^{\sqrt{u}(1+x_D)}I_0(z)dz\right] \quad (11)$$

The "ascending" series form of the $I_0(z)$ is given in Abramowitz and Stegun[3] as:

$$I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \dots \qquad (12)$$

The third and higher order terms may be neglected as $z \to 0$. By using Eq. (12), the integration terms of $I_0(z)$ in Eq. (11) may be reduced to:

$$\int_0^{\sqrt{u}(1-x_D)} I_0(z)dz + \int_0^{\sqrt{u}(1+x_D)} I_0(z)dz = 2\sqrt{u} + \frac{u\sqrt{u}}{6} + \frac{u\sqrt{u}x_D^2}{2} \qquad (13)$$

From Eq. (13), and using the binomial series form for $(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots$ for $|z| < 1$, the $I_0(\sqrt{u}r_{eD})$ term may be written as:

$$\frac{1}{I_0\left(\sqrt{u}r_{eD}\right)} = 1 - \frac{\left(\sqrt{u}r_{eD}\right)^2}{4} \qquad (14)$$

So, the simplified equation in Laplace domain is:

$$\bar{p}_{D,cpb}(u, r_{eD}, x_D) = \bar{p}_{D,inf}(u, x_D) - K_0\left(\sqrt{u}r_{eD}\right)\left[\frac{1}{12} + \frac{1}{u} + \frac{x_D^2}{4} - \frac{r_{eD}^2}{4} - \frac{ur_{eD}^2}{4}\left(\frac{1}{12} + \frac{x_D^2}{4}\right)\right] \qquad (15)$$

The $\frac{ur_{eD}^2}{4}\left(\frac{1}{12} + \frac{x_D^2}{4}\right)$ term may be neglected as $\sqrt{u}r_{eD} \to 0$. The final form of the Laplace domian solution is:

$$\bar{p}_{D,cpb}(u, r_{eD}, x_D) = \bar{p}_{D,inf}(u, x_D) - K_0\left(\sqrt{u}r_{eD}\right)\left[\frac{1}{12} + \frac{1}{u} + \frac{x_D^2}{4} - \frac{r_{eD}^2}{4}\right] \qquad (16)$$

This equation can be inverted to the real domain by using the inversion table. The resulr is:

$$p_{D,cpb}\left(t_{LfD}, r_{eD}, x_D\right) = p_{D,inf}\left(t_{LfD}, x_D\right) - \frac{1}{2}E_1\left(\frac{r_{eD}^2}{4t_{LfD}}\right) - \frac{1}{8t_{LfD}}\left[\frac{1}{3} + x_D^2 - r_{eD}^2\right]\exp\left(\frac{-r_{eD}^2}{4t_{LfD}}\right) \qquad (17)$$

The "well testing" derivative form of Eq. (17) is:

$$p'_{D,cpb}\left(t_{LfD}, r_{eD}, x_D\right) = p'_{D,inf}\left(t_{LfD}, x_D\right) - \frac{1}{2}\exp\left(\frac{-r_{eD}^2}{4t_D}\right) + \left[\frac{1}{3} + x_D^2 - r_{eD}^2\right]\left(\frac{1}{8t_D} - \frac{r_{eD}^2}{32t_D^2}\right)\exp\left(\frac{-r_{eD}^2}{4t_D}\right) \qquad (18)$$

- No-Flow Outer Boundary Case

The outer boundary condition for a no-flow boundary case is:

$$\left(r_D\frac{\partial p_D}{\partial r_D}\right)_{r_D = r_{eD}} = 0 \qquad (19)$$

The Laplace domain solution is given by Okan and Ranghavan[2] as:

$$\bar{p}_{D,nfb}(u, r_{eD}, x_D) = \bar{p}_{D,inf}(u, x_D) + \frac{K_1\left(\sqrt{u}r_{eD}\right)}{2uI_1\left(\sqrt{u}r_{eD}\right)}\left[\int_0^{\sqrt{u}(1-x_D)} I_0(z)dz + \int_0^{\sqrt{u}(1+x_D)} I_0(z)dz\right] \qquad (20)$$

The $\frac{K_1(z)}{I_1(z)}$ term may be inverted into simple known functions by using the Wronskian relation for modified Bessel functions as:

$$\frac{K_1(z)}{I_1(z)} = \frac{K_2(z) - K_0(z)}{I_0(z) - I_2(z)} \qquad (21)$$

The "ascending" form of $I_1(z)$ can be written as:

$$I_1(z)=\frac{z}{2}+\frac{z^3}{16}+\frac{z^5}{386}+\ldots \qquad (22)$$

The third and higher order terms can be neglected as $z \to 0$. By using the fact that the third and higher order terms in $(1+z)^{-1}=1-z+z^2-z^3+\ldots$ for $|z|<1$ can be neglected as $z \to 0$, the reciprocal of $I_1(z)$ may be written as:

$$\frac{1}{I_1(z)}=\frac{2}{z}\left(1-\frac{z^2}{8}\right) \qquad (23)$$

The $I_0(z)-I_2(z)$ term may be written in terms of $I_1(z)$ as:

$$I_0(z)-I_2(z)=\frac{2}{z}I_1(z) \qquad (24)$$

Using these relations, Eq. (21) may be reduced to yield:

$$\frac{K_1(z)}{I_1(z)}=\left(1-\frac{z^2}{8}\right)\left[K_2(z)-K_0(z)\right] \qquad (25)$$

So, the final form in the Laplace domain may be written as:

$$\bar{p}_{D,nfb}(u,r_{eD},x_D)=\bar{p}_{D,inf}(u,x_D)+\frac{1}{2u}\left(1-\frac{ur_{eD}^2}{8}\right)\left\{K_2(z)-K_0(z)\right\}\left[2+\frac{u}{12}\left(2+6x_D^2\right)\right] \qquad (26)$$

By neglecting some terms as $\sqrt{u}r_{eD} \to 0$, Eq. (26) may be reduced to:

$$\bar{p}_{D,nfb}(u,r_{eD},x_D)=\bar{p}_{D,inf}(u,x_D)+\left[\frac{1}{u}+\frac{1}{12}\left(1+3x_D^2\right)-\frac{r_{eD}^2}{8}\right]\left\{K_2(z)-K_0(z)\right\} \qquad (27)$$

Using standard tables of Laplace transforms, we can invert Eq. (27) term-by-term to yield:

$$p_{D,nfb}\left(t_{LfD},r_{eD},x_D\right)=p_{D,inf}\left(t_{LfD},x_D\right)-\frac{1}{2}\mathrm{E}_1\left(\frac{r_{eD}^2}{4}\right)+\frac{1}{r_{eD}^2}\left(\frac{1}{6}+\frac{x_D^2}{2}+2t_D-\frac{r_{eD}^2}{4}\right)\exp\left(\frac{-r_{eD}^2}{4t_{LfD}}\right) \qquad (28)$$

and the derivative form is:

$$p'_{D,nfb}\left(t_{LfD},r_{eD},x_D\right)=p'_{D,inf}\left(t_{LfD},x_D\right)+\left[\frac{1}{4t_{LfD}}\left(\frac{1}{6}+\frac{x_D^2}{2}-\frac{r_{eD}^2}{4}\right)+\frac{2t_{LfD}}{r_{eD}^2}\right]\exp\left(\frac{-r_{eD}^2}{4t_{LfD}}\right) \qquad (29)$$

• Construction of Type Curves

In this paper, the exact real domain from Laplace domain is determined by using Gaver-Stehfest algorithm because most of the solutions developed in the Laplace domain are too complicated for inversion using techniques of complex analysis. The Gaver-Stehfest algorithm is given as:

$$f_a(t)=a\sum_{i=1}^{n}v_i\bar{f}(ai) \qquad (30)$$

where:

$$a=\frac{\ln(2)}{t} \qquad (31)$$

and the Stehfest coefficients are given by:

$$v_i = (-1)^{(n/2)+i} \sum_{k=\frac{(i+1)}{2}}^{\min\left(i,\frac{n}{2}\right)} \frac{k^{n/2}(2k)!}{\left(\frac{n}{2}-k\right)!k!(k-1)!(i-k)!(2k-i)!}, i = 1,2,\ldots,n \quad (32)$$

which is the correct form[4]. The Gaver-Stehfest algorithm usually gives acceptable results for a wide range of problems using $4 < n < 20$, where $n$ is always even. In this work, $n = 6$ is used. Low values of $n$ should be used for complicated problems.

## 3. VERIFICATION

Figures 1 and 2 are log-log plots of an analytical, numerical inversion of Eq. (10), and approximate dimensionless pressure Eq. (17) and pressure derivative solutions Eq. (18) for the uniform flux and infinite conductivity vertical fracture cases with a constant pressure outer boundary reservoir, respectively. The plot show good agreement between the analytical, and approximate values of dimensionless pressure. There is some disagreement for dimensionless pressure derivative which may be the result of a drastic decrease in dimensionless pressure derivative values as the dimensionless pressure function becomes constant.

The comparison between Eqs. (20) and (28), and their derivatives is presented in Figures 3 and 4 for a uniform flux and an infinite conductivity vertical fracture, respectively. There is an good agreement between the analytical and approximate dimensionless pressure and pressure derivative.

From observing, the behavior of the derivative approximation for both cases is in good agreement, except in the region where the reservoir approach steady-state flow conditions. Other solutions with more complex expansion do not give better performance than the cases presented in this paper. It is also noticeable that the inner boundary has little effect on the pseudo-steady state region.

## 4. SUMMARY AND CONCLUSIONS

The approximate solutions are developed for the Laplace transform and real space solutions for a fractured well with a constant flow rate centered in a bounded circular reservoir with a no flow or constant pressure outer boundary.

Each of these solutions has been shown to be accurate compared to its analytical solutions, the numerical inversion of the Laplace transform solution. These relations may be used to develop theoretical results and analysis relations.

## 5. ACKNOWLEDGEMENT

We would like to thank Texas A&M U. and Chulachomklao Royal Military Academy for all computing resource. We also would like to thank Col.Prasart Sukkasate for his comment.

## 6. NOMENCLATURE

| | | |
|---|---|---|
| $B$ | = | formation volume factor, Reservoir Barrel per Stock Tank Barrel |
| $c_t$ | = | total system compressibility, $psi^{-1}$ |
| $C_{fD}$ | = | $\frac{0.8936C}{\phi c_t h L_f^2}$ dimensionless fracture storage coefficient |
| $k$ | = | formation permeability, md |
| $p_D$ | = | $\frac{kh}{141.2qB\mu}\Delta p$, dimensionless pressure function for the constant flow rate case |
| $p_{fD}$ | = | dimensionless formation flow rate |
| $p'_D$ | = | $t_D\frac{\partial p_D}{\partial t_D}$, logarithmic derivative of dimensionless pressure function for the constant flow rate case |
| $\overline{p}_D$ | = | Laplace transform of dimensionless pressure |
| $q$ | = | surface flow rate, Stock Tank Barrel per day |
| $q_D$ | = | $\frac{qB\mu}{141.2kh(p_i - p_{wf})}$ dimensionless flow rate |
| $q_{fD}$ | = | dimensionless formation flow rate |
| $r_D$ | = | $\frac{r}{r_w}$ dimensionless radius |
| $r_{eD}$ | = | $\frac{r}{r_e}$ dimensionless drainage radius of the reservoir |
| $r_e$ | = | external radius |
| $r_w$ | = | wellbore radius |
| $s$ | = | skin factor |
| $t$ | = | time, hrs |
| $t_{LfD}$ | = | $0.0002637\frac{kt}{\phi\mu c_t L_f^2}$, dimensionless time based on fracture half-length |
| $u$ | = | Laplace space variable, dimensionless |
| $x_D$ | = | $\frac{x}{x_f}$ dimensionless distance in $x$ direction |
| $x_f$ | = | Fracture half-length |
| $z$ | = | variable |
| $\phi$ | = | porosity, fraction |

$\mu$ = viscosity, cp

## REFERENCES

1. Gringarten, A. C., Ramey, H. J., Jr. and Raghavan, R.: "Unsteady-state Pressure Distributions Created by a Well with a Single Infinite-Conductivity Vertical Fracture," *SPEJ,* (August 1974) 347-359.
2. Ozkan, O. and Raghavan, R.: "New Solutions for Well-Test-Analysis Problems: Part 2 -- Analytical Considerations," *SPEFE* (September 1991) 359-368.
3. Abramowitz, M. and Stegun, I.: *Handbook of Mathematical Functions*, Dover Publications, New York (1972).
4. Stehfest, H.: "Numerical Inversion of Laplace Transforms," *Communications of the ACM* (January 1970), 13, No. 1, 47-49. (Algorithm 368 with correction)