การแก้สมการเบอร์เกอร์ด้วยวิธีไฟไนท์ดิฟเฟอร์เรนท์และการวิเคราะห์เวฟเลทประยุกต์

Finite Difference Solutions to The Burgers Equation and Applications to Wavelet Analysis

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ผลเฉลยของสมการเบอร์เกอร์ได้แสดงโดยวิธีไฟไนท์ดิฟเฟอร์เรนท์ ร่วมกับเทคนิคการแบ่งช่วงเวลาเพื่อแยกสมการเบอร์เกอร์ออกเป็นสองส่วนคือ ส่วนที่เป็นสมการไม่เป็นเชิงเส้นและส่วนที่เป็นสมการเชิงเส้น การแก้สมการที่ไม่เป็นเชิงเส้นใช้ 4 วิธี โดยสองวิธีแรกเป็นเอ็กซ์พรีสิทคือ Adam- Bashforth (AB)และ Quadratic Upstream Interpolation Convective Kinematics (QUICK) สองวิธีหลังเป็นการทำให้สมการเป็นเชิงเส้นด้วยวิธี Newton's linearization (NL) และ Richtmyer linearlization (RL) สำหรับสมการเชิงเส้นเราใช้วิธี Crank - Nicolson (CN) ซึ่งเป็นวิธีที่ไม่มีเงื่อนไขการมีเสถียรภาพเมื่อรวมเทคนิคดังกล่าวเข้าด้วยกันจะได้การแก้สมการเบอร์เกอร์ 4 วิธีคือ AB/CN , QUICK/CN , NL/CN และ RL/CN การวิเคราะห์คำตอบที่ได้จากวิธีต่าง ๆ นั้นทำทั้งบน Spatial domain และ Wavelet domain ซึ่งพบว่าการวิเคราะห์บน Wavelet domain นั้นมีประโยชน์ มากโดยสามารถวิเคราะห์ ผลเฉลยได้บน สเกลความยาว(Length scale)ที่ต่างกันซึ่งจะให้ความหมายทางกายภาพอย่างชัดเจน

Solutions to the Burgers equation are determined from finite difference schemes. The time splitting technique is used to bisect the Burgers equation into a non-linear convection and a linear dissipation parts. Four finite difference techniques are used to solve the non-linear part. The first two are explicit techniques: a third order Adams-Brashford (AB) technique and the Quadratic Upstream Interpolation for Convective Kinematics (QUICK) technique. The other two are linearization techniques: Newton's (NL)and Richtmyer's (RL) linearization techniques. On the other hand, Crank-Nicolson (CN) scheme (an unconditional stable scheme) is used to solve the linear dissipation part. Combinations of these give four FD schemes for solving the Burgers equation : AB/CN , QUICK/CN , NL/CN , and RL/CN. Analysis of the FD-solutions based on spatial domain and their projection onto wavelet domain have been done. It has been found that analyzing the solutions in the wavelet domain is very powerful ; it gives a clear physicsal behavior of the solutions by decomposing the solution onto different length scale .

1. Introduction

Dissipation is one of the turbulent flow characteristics that occurs simultaneously at different length scales[1]. To study this characteristic, we simplify 3-D dynamic equations of fluctuating velocity components into a well-known 1-D Burgers equation. Although fictitious, the solution represents an interesting time-varying multi-scale energy dissipation[2].

While different schemes have been applied to solve the Burgers equation [3,4], analyses of the solutions have been focused mainly on comparing both the maximum value of slopes and the corresponding time with the values obtained from the exact solution occurring at a particular point. Although the approach was quantitatively reasonable, the physical qualities of the solutions were obscure.

In this paper, we use finite difference methods to solve the Burgers equation and propose wavelet technique to analyze the solutions, The solution method begins with time splitting technique[5] so that the Burgers equation can be separated into a nonlinear-convection and a linear-dissipation equations. The nonlinear part is solved by four different techniques; Adams-Brashford(AB)[4], Quadratic Upstream Interpolation for Convective Kinematics (QUICK)[6], Newton's and Richtmyer's linearlization (NL and RL)[7] while the linear dissipation equation is solved solely by Crank-Nicolson(CN) scheme[7]. A unique solution is obtained from all schemes when fine enough grid points are used .

Analyses of the solutions by discrete wavelets transform are shown in section 4 . The wavelet-analysis decomposes the solution into detailed functions of different length scales. This localizes the sharp variation both in the wavelets and spatial domains and allows the solution to be analyzed at each scale independently so that multi-scale dissipation behaviors of the Burgers solution become evident .

2. Formulation

The Burgers equation, its initial and boundary conditions are as follow

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial^2 x},\ |x|\le 1,\ t>0$$

$$u(1,t)=u(-1,t)=0 \qquad (1)$$

$$u(x,0)=-sin(\pi x)$$

Here the boundary values are fixed at zero while the sine wave represents an initial disturbance. We use time splitting technique to separate eqn.(1) into a linear and nonlinear partial differential equations, hence

$$\frac{1}{2}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \qquad (2)$$

$$\frac{1}{2}\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial x^2} \qquad (3)$$

Note that combination of eqn.(2) and eqn.(3) results in the original Burgers equation. The splitting technique allows us to solve the nonlinear term in the Burgers equation separately. This is done by solving eqn.(2) for an intermediate solution $\hat{u}$. Use $\hat{u}$ as an initial condition to eqn.(3) and determine the required solution u. We apply finite difference discretizations to both eqns.(2) and (3) . As mentioned earlier, four different schemes; AB, QUICK, NL and RL, are employed to solve eqn.(2) for $\hat{u}$ then the unconditionally stable CN scheme is used to determine u, applying $\hat{u}$ as the initial condition. All finite difference discretization formulae summarized in appendix A.

3. The Burgers Solutions

The Burgers solutions for two different values of ν(1/4π and 1/100π) are shown respectively in Figs. 1 and 2. For higher viscosity values, the diffusion term dominates and the gradient of the solution at the origin is weak. Conversely the solution exhibits a sharp variation when viscosity is small .

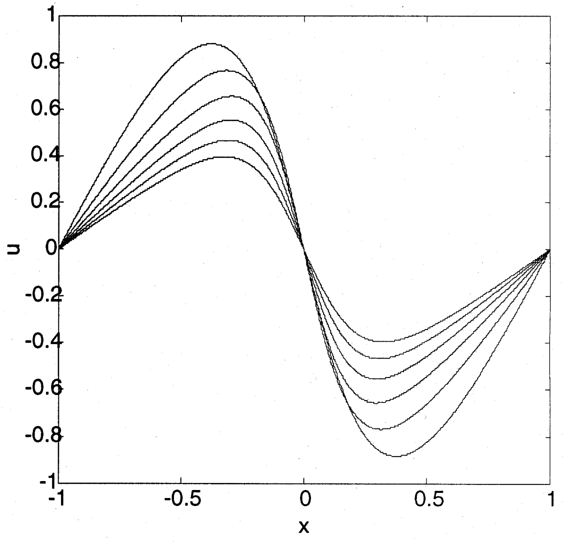

Fig1. : Solution of Burgers equation with viscosity (1/4π) at various time (t = 0.5/π , 1/π , 1.5/π , 2/π , 2.5/π, 3/π) using AB/CN 1024 grid points

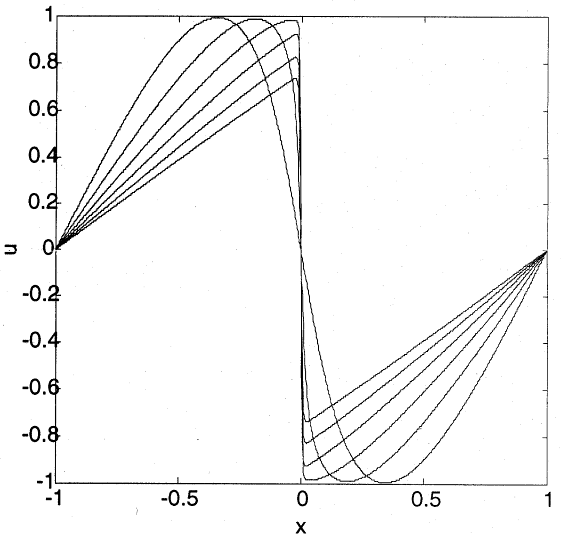

Fig2. : Solution of Burgers equation with viscosity (1/100π) at various time (t = 0.5/π , 1/π , 1.5/π , 2/π , 2.5/π, 3/π) using AB/CN 1024 grid points

An attempt is made to evaluate the relative merits of each of the four numerical schemes chosen by calculating the maximum value $\left|\frac{\partial u}{\partial x}\right|_{x=0}$ and the corresponding time($t^*_{\max}$).For example, it has been found that $\left|\frac{\partial u}{\partial x}\right|_{\max}$ and $t^*_{\max}$ are 155.497 and 0.509 respectively for the AB/CN schemes(Fig.3). We then compare the results with the values obtained from the exact solution The results from all schemes are summarized in table 1. We can conclude from the values of $\left|\frac{\partial u}{\partial x}\right|_{\max}$ that at least 512 grid points have to be included in the spatial discretizations in order to resolve the sharp variations. The time t_{max} does not change much from one method to the other since we are oversampling in time (Δt = 1/1000π) and that all values obtained are good approximations .

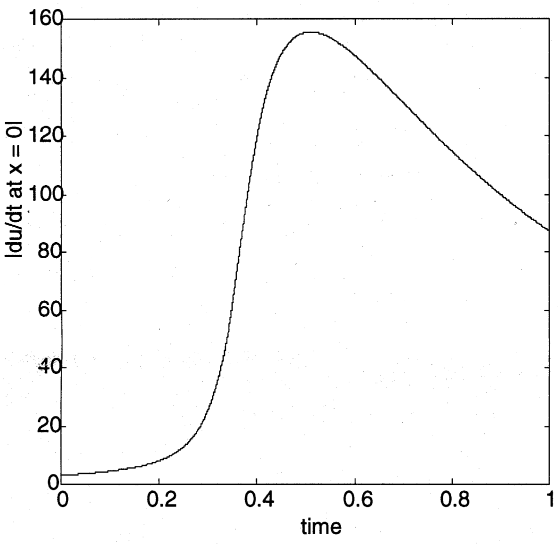

Fig3. : $\left|\frac{\partial u}{\partial x}\right|_{x=0}$ against time (viscosity = 1/100π , AB/CN , 1024 grid points)

Table 1 : Maximum absolute value of slope at origin and time of it occurrence for each numerical scheme and comparison with analytical solution

method	grid	$\left\|\frac{\partial u}{\partial x}\right\|_{x=0,\max}$	$t^*_{\max}$
AB/CN	1024	155.49775	0.509529
	512	148.68965	0.509741
	256	127.27110	0.510153
QUICK/CN	1024	155.67638	0.511206
	512	150.37589	0.510569
	256	139.02521	0.510251
NR/CN	1024	155.26184	0.511206
	512	148.33331	0.510569
	256	126.90212	0.510251
RY/CN	1024	154.50588	0.509614
	512	147.91867	0.509932
	256	126.87027	0.510251
Analytical		152.00516	0.51047

It is evident that this maximum slope-time comparison technique, although giving a quantitative values of the maximum slope and the corresponding time of occurrence, does not provide any physical meaning of the solutions. In the next section we will project the solutions onto the wavelet domain and analyze the wavelet coefficients instead.

4.Discrete Wavelet Transforms

In this paper we use Daubechies wavelet as our basis function for analyzing the Burgers solutions. The wavelet is defined by [8]

$$W(x) = -c_3\phi(2x) + c_2\phi(2x-1) - c_1\phi(2x-2) + c_0\phi(2x-3) \tag{4}$$

where $\phi(x)$ is a scaling function. The scaling function is iteratively calculated from a dilation equation of the form

$$\phi(x) = c_o\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3) \tag{5}$$

where a particular set of the coefficients defining Daubechies wavelet and the scaling function is

$$c_o = (1+\sqrt{3})/4, c_1 = (3+\sqrt{3})/4,$$
$$c_2 = (3-\sqrt{3})/4, \text{and } c_3 = -(\sqrt{3}-1)/4.$$

Subject to a few conditions [9], any arbitrary signal f(x) can be decomposed into an infinite summation of wavelets at different scales according to the expansion

$$f(x) = \sum_{k=-\infty}^{\infty} c_{\phi,k}\phi(x-k) + \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} c_{j,k}W(2^j x - k). \tag{6}$$

When the function f(x) is periodic and is discretized into 2^J points at equally spaced intervals over the period $0 \leq x < 1$, there exists a discrete wavelet transform (DWT) algorithm called Mallat's pyramid algorithm that transforms $f(x_n)$ into 2^J transform coefficients ($a_0, a_1, a_2, \ldots, a_{2^J-1}$). The function f(x) can then be approximated from the set of the transformed coefficients, by,

$$f(x) = a_o\phi(x) + a_1W(x) + a_{2^1}W(2^1x) + a_{2^1+1}W(2^1x-1) + \ldots + a_{2^{J-1}}W(2^{J-1}x) + \ldots + a_{2^{J-1}+2^{J-1}-1}W(2^{J-1}x - 2^{J-1} - 1). \tag{7}$$

Note that there are 2^j transform coefficients at the j^{th} scale.

Inverse discrete wavelet transform (IDWT) of the coefficients at each scale returns detailed function of $f(x_k)$ at that scale(see Fig.4). Also, since the wavelet W(x) is compactly supported and localized at x = 0, the wavelet transform coefficient a_{2^j+k} represent the strength of the wavelet component of the scale j at the point k. The capability of scale decomposition and local transformation behaviors make wavelet transforms well-suited to analyze the time varying multi-scale dissipation Burgers solution.

In Fig.4, the left column shows reconstruction results of the detailed functions at different levels. Combination of these functions at lower levels gives large scale solution to the Burgers' solution. At higher levels, the detailed functions are confined locally within the region where Burgers solution exhibits sharp variation. These mean that, the ensemble of these detailed functions at higher levels does not affect large scale solution. Therefore, if details of the sharp variation are not required, the Burgers solution could be reconstructed from the combination of only first few levels of detailed functions.

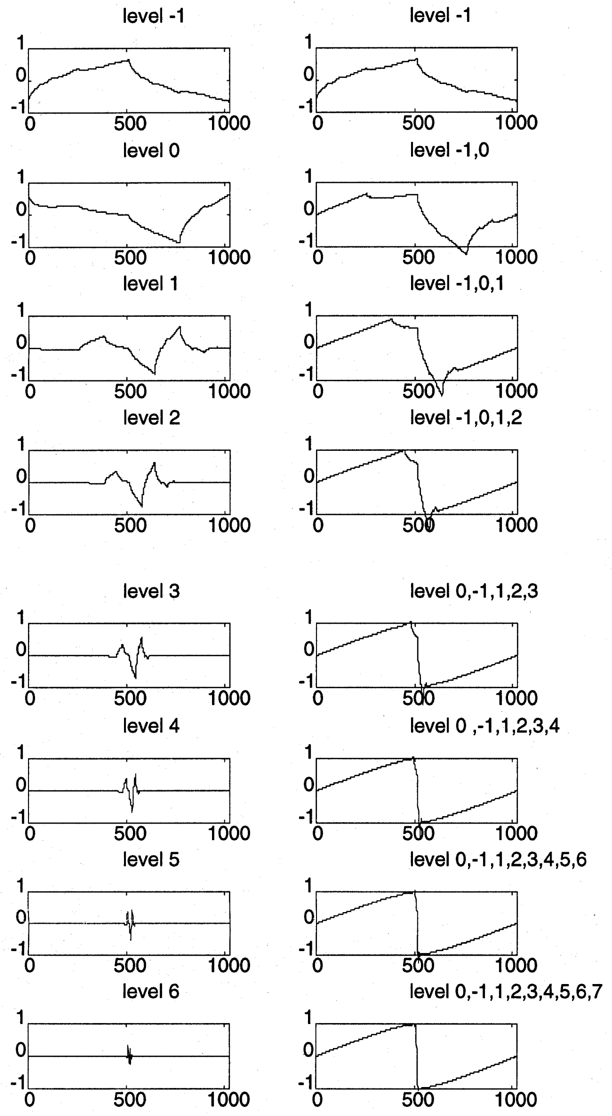

Fig.4 : Reconstruction of solution of Burger's equation, the left column shows detailed functions and the right column shows combination of detailed function.

Effects of spatial resolution on the accuracy of the finite difference Burgers solutions from AB/CN, RL/CN and QU/CN methods are shown in Fig.5. The calculation result from NL/CN method is omitted from the plot since it has a similar result to RL/CN method. It is shown in Fig.5 that the norms of wavelet coefficients at each scale converges to a unique value when the spatial grid points are fine enough. The smaller scale solutions, however, converge slower than large scale solutions. Thus the errors are from small scale solutions and in this particular problem are localized at the region where the small scale detailed functions are located .

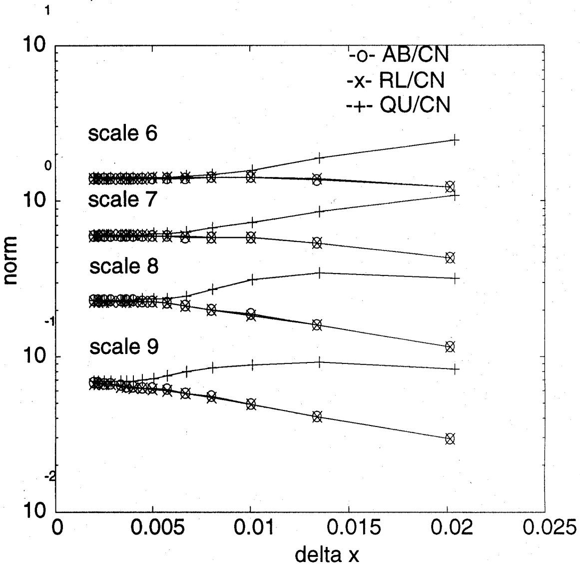

Fig.5 : Wavelet-norm of each scale vs. Δx for each numerical schemes

The dissipation characteristic of the Burgers' solutions can also be analyzed by considering variations of wavelet norms with time. Figs.6 and 7 show multi-scale plot of norms with time for high ($1/4\pi$) and low ($1/100\pi$) viscosity respectively. At the largest scale, the energy decreases continuously from the beginning while at other lower scales the energy begins to decrease at later time. The effect becomes evident when viscosity is small (see Fig.7). However, to the left of the cross lines in both Figs.6 and 7, energy of all scales are monotonously dissipated. This means that the magnitude of viscosity does not only affect the rate of dissipation but also affects the distribution of energy among different length scales.

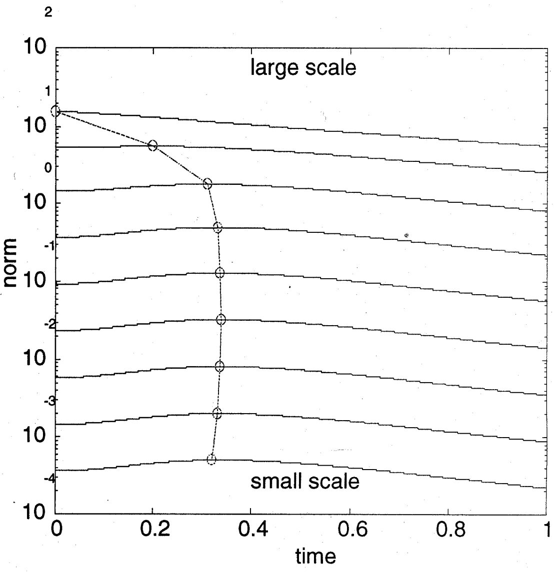

Fig.6 : Wavelet -norms vs time at different length scales ($\nu=1/4\pi$)

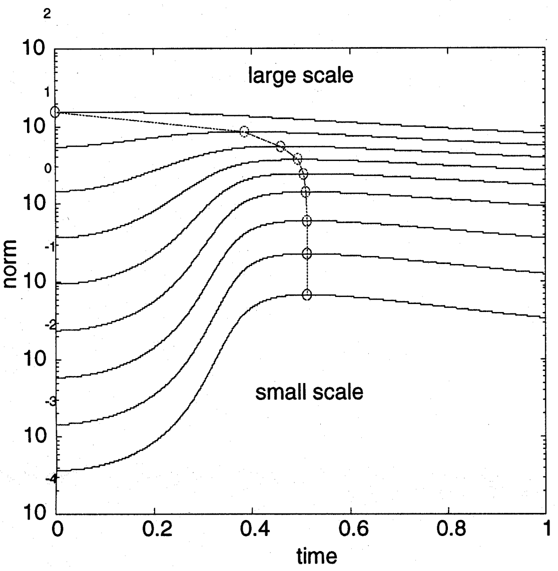

Fig.7 : Wavelet -norms vs time at different length scales (ν=1/100π)

5.Conclusions

Solving the Burgers equation by four different finite difference schemes results in a unique solution provided that fine enough grid points are used. Analyzing the solutions with the wavelets transform has a major advantage in that the solutions can be decomposed into detailed functions of different length scales. This allows spatial and temporal analyses to be done scale by scale resulting in clear representations of physical effects of viscosity coefficient on the Burgers solution. Also, the results make us believe that wavelet transform is a good tools for analyzing complex flows containing complex multi-scale interactions, e.g. turbulent flow.

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Appendix A.

Numerical Schemes

Convection part

1. 3rd Adams-Bashforth(AB)

$$\frac{\hat{u}-u^n}{\Delta t} = -\frac{1}{4\Delta x}\left(\frac{23}{12}\delta_x(u^n)^2 - \frac{16}{12}\delta_x(u^{n-1})^2 + \frac{5}{12}\delta_x(u^{n-2})\right)$$

which $\delta_x u_i^n = u_{i+1}^n - u_{i-1}^n$

Note : The first two step begin Euler forward

2. QUICK

$$\frac{\hat{u}-u^n}{\Delta t} = \frac{1}{2\Delta x}((u_l)^2 - (u_r)^2)$$

where

$$u_r = \frac{1}{2}(u_{i+1}+u_i) - \frac{1}{8}\Delta x^2 curv_r$$

$$u_l = u_r(i-1)$$

$$curv_r = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}, \text{ if } u_r \geq 0$$

$$= \frac{u_{i+2} - 2u_{i+1} + u_i}{\Delta x^2}, \text{ if } u_l \geq 0$$

Note : u_r, u_l are velocity at wall as interpreted as a linear interpolation between grid corrected by term proportional to stream curvature

$curv_r, curv_l$ are corrected term of stream curvature

3. Crank-Nicolson with Newton Ralphson linearization(NR)

$$\frac{\hat{u}-u^n}{\Delta t} = -\frac{1}{8\Delta x}\left(\delta_x \hat{u}^2 + \delta_x (u^n)^2\right)$$

4. Crank-Nicolson with Ritchmyer's linearization(RY)

$$\frac{c}{4}\delta_x\left(u^n\omega\right) + \omega = \delta_x(u^n)^2$$

where $\omega_i = \hat{u} - u^n$

Diffusion part

Linear diffusion

Crank-Nicolson(CN)

$$\frac{u^{n+1} - \hat{u}}{\Delta t} = \frac{1}{2\Delta x^2}\nu\left(\delta_x^2 u^n + \delta_x^2 \hat{u}\right)$$

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