TSME ICOME 2016
Mechanical Engineering Innovation for Green Society

AMM0007

Motion of a Particle on a Plate Rotating with a Constant Speed

Niranad Chaisiri[1,*], and Lerkiat Vongsarnpigoon[1]

[1] Faculty of Engineering, Thai-Nichi Institute of Technology,
Pattanakarn Road, Suanluang, Bangkok 10250, Thailand
[*] ch.niranad_st@tni.ac.th

Abstract

This paper studies the motion of a body placed on a flat surface which is rotating at a constant speed. Because of the difference in velocity at the contact point, slipping will initially occur. There are two possibilities for the subsequent motion: either the body stops moving with respect to the plate or it keeps on slipping and moves out of the surface. Equations of motion for the body are written down using a polar coordinate system and solved numerically for various initial conditions of rotating speed and the initial radial distance. It is found that a clear boundary can be established separating the region of initial conditions for which the body will stop relative to the plate from that for which it will not. The initial conditions are represented by the rotating speed ω and a parameter $G = \mu_k g / r_0$, where μ_k is the kinetic coefficient of friction, g the gravitational constant, and r_0 the initial radial distance.

Keywords: rotating plate, relative velocity, slipping.

1. Introduction

Rotating surfaces are common both in everyday lives and in industries, e.g. turntables, grinding surfaces, merry-go-rounds, etc. The problem of a body sitting on a flat rotating surface is considered either as an example or exercise in most basic Engineering Mechanics books [1,2]. The problem is usually to find a critical rotating speed such that a particle sitting on the surface with a given coefficient of friction will not slip. The question of how the body comes to be at that particular place has rarely been considered. Of course, one can imagine a mechanism to hold the body in place while the plate increases its speed or to match the speed of the plate before placing the body on it. It is more interesting to ask if this is possible without any external mechanism.

Vongsarnpigoon and Sratong-on [3] considered the problem of a plate starting from rest with a body on it and asked how the plate could be brought up to the critical speed of rotation without the body slipping. It was found that there were several ways of achieving the critical speed. For example, the plate could start out at a maximum allowable acceleration and decreased its acceleration linearly until it became zero at the critical speed. Some scheme of motion is shown to be impossible since the body would always slip before the plate reaching critical speed, e.g. a motion with constant acceleration no matter how small.

A similar and related problem is to consider a flat plate already rotating at a constant angular speed with a body suddenly placed on top of it. This is the subject of this paper.

2. Basic equations

Consider a body with mass m being place on a flat plate which is rotating at a constant angular velocity ω at a distance r_0 from the center of the plate. Using a cylindrical polar coordinate system (r, θ) and treating the body as a particle, the velocity and acceleration vectors of the particle at any time is given by

$$\vec{\boldsymbol{v}}_b = \dot{r}\vec{\boldsymbol{e}}_r + r\dot{\theta}\vec{\boldsymbol{e}}_\theta \ ,$$
$$\vec{\boldsymbol{a}}_b = (\ddot{r} - r\dot{\theta}^2)\vec{\boldsymbol{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{\boldsymbol{e}}_\theta \ , \quad (1)$$

where $\vec{\boldsymbol{e}}_r$ and $\vec{\boldsymbol{e}}_\theta$ are the base vectors along the r- and θ-direction, respe
ctively, and the superposed dot denote a derivative with respect to time t.

The equations of motion for the body are, therefore, given by

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= f_r \ , \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= f_\theta \ , \end{aligned} \quad (2)$$

where m is the mass of the body, f_r and f_θ are, respectively, the friction force in the r- and θ-directions. Since the surface is flat, the normal reaction from the surface and the maximum friction force are

$$\begin{aligned} N &= mg \ , \\ f_{\max} &= \mu_s mg \ , \end{aligned} \quad (3)$$

where μ_s is the static coefficient of friction between the body and the surface, and g is the gravitational constant. Before going further, it should be noted here that in many basic Engineering Mechanics textbook [1,2], Eq. (2) lead to a critical angular velocity ω_{cr} at which the body can remain on the plate without slipping. If the body is stationary with respect to the surface, the relative velocity becomes zero and $\dot{\theta} = \omega$, Eq. (2) reduce to

$$-r\dot{\theta}^2 = \frac{f_r}{m} \ ,$$
$$r\ddot{\theta} = \dot{r} = \ddot{r} = 0 \ . \qquad (4)$$

With $f_r = f_{\max} = \mu_s mg$, the angular velocity becomes the critical angular velocity ω_{cr}, then

$$\omega_{cr}{}^2 = \frac{\mu_s g}{r} = G_1 \ , \qquad (5)$$

where G_1 is a parameter identical to the parameter c^2 in [3].

When a body is initially placed on the spinning plate at a radial distance r_0, its real velocity is zero but the relative velocity of the body with respect to the plate is nonzero; hence, slipping occurs. The force acting on the body in the horizontal direction is the kinetic friction force. The velocity of the rotating plate at any point r from the center is given by

$$\vec{\boldsymbol{v}}_p = r\omega\vec{\boldsymbol{e}}_\theta \ . \qquad (6)$$

With Eqs. (1) and (6), the relative velocity of the body with respect to the plate can be written as

$$\vec{\boldsymbol{v}}_{b/p} = \dot{r}\vec{\boldsymbol{e}}_r + r(\dot{\theta}-\omega)\vec{\boldsymbol{e}}_\theta \ . \qquad (7)$$

Since the friction force is always in the opposite direction to the relative velocity, it has the form

$$\vec{\boldsymbol{f}} = \mu_k mg\left(\frac{-\dot{r}\vec{\boldsymbol{e}}_r + r(\omega-\dot{\theta})\vec{\boldsymbol{e}}_\theta}{\sqrt{(-\dot{r})^2 + r^2(\omega-\dot{\theta})^2}}\right) \ , \qquad (8)$$

where μ_k is the kinetic coefficient of friction. The equations of motion (2) become

$$\ddot{r} - r\dot{\theta}^2 = \mu_k g\frac{(-\dot{r})}{\sqrt{(-\dot{r})^2 + r^2(\omega-\dot{\theta})^2}} \ ,$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \mu_k g\frac{r(\omega-\dot{\theta})}{\sqrt{(-\dot{r})^2 + r^2(\omega-\dot{\theta})^2}} \ . \qquad (9)$$

Eq. (9) above is a system of nonlinear ordinary differential equations which cannot be solved analytically and a numerical method is needed. Eq. (9) describes the motion of the body on the surface in terms of its position r and θ as long as slipping occurs, i.e. the body is moving with respect to the surface. In some cases, the body will stop with respect to the surface, and the relative velocity in Eq. (7) vanishes.

3. Simulations

For the purpose of numerical calculation, define a non-dimensional radial variable R and a new variable Ω as follows:

$$R = \frac{r}{r_0} \ ,$$
$$\Omega = \dot{\theta} \ . \qquad (10)$$

Then, Eq. (9) can be rewritten as

$$\ddot{R} - R\Omega^2 = G\frac{(-\dot{R})}{\sqrt{(-\dot{R})^2 + R^2(\omega-\Omega)^2}} \ ,$$
$$\dot{\Omega} + 2\dot{R}\Omega = G\frac{R(\omega-\Omega)}{\sqrt{(-\dot{R})^2 + R^2(\omega-\Omega)^2}} \ , \qquad (11)$$

where G is a parameter defined by

$$G = \frac{\mu_k g}{r_0} \ . \qquad (12)$$

It should be noted that G is a combination of the physical characteristics of the contact surfaces and the initial radial distance r_0 and has the dimension of s^{-2}. Also, the initial conditions of Eq. (11) are given by

$$R(0) = 1 \ , \ \Omega(0) = 0 \ . \qquad (13)$$

The numerical calculations are carried out using 4th-order Runge-Katta method [4] run on MATLAB. With a given value of ω, a starting value of G is chosen. The routine for solving Eq. (11) is then run and the relative radial velocity and the relative angular velocity of the body with respect to the plate are recorded. A typical result in which the body stops with respect to the plate is shown in Fig. 1. In this case, the angular velocity of the plate is $\omega = 20$ rad/s, and the initial relative angular velocity is therefore equal to 20 while the relative radial velocity is 0. As time progresses, both relative velocities change and reach 0 simultaneously. This means the body stops moving with respect to the plate from then on.

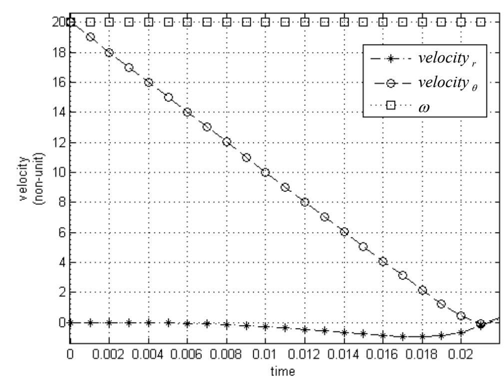

Fig. 1 Relative velocity in the r- and θ-directions for a case which the relative motion stops

In some cases, the relative radial and angular velocities do not approach zero which means the body continues to move with respect to the plate. A typical result of a non-stopping body is shown in Fig. 2. In this case, the angular velocity of the plate is 50 rad/s. The relative angular velocity first decreases but then reverses direction and increases continuously while the relative radial velocity decreases monotonically. It is clear that the particle cannot stop relative to the plate.

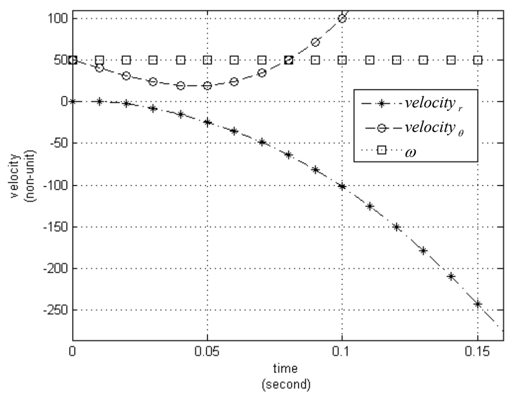

Fig. 2 Relative velocity in the r- and θ-directions for a case which the relative motion does not stop

For a given ω, it is found that there is a limiting value of G, i.e. a value of G below which the body would not stop. The plot of limiting values of G versus ω is shown in Fig. 3. As can be seen, the limiting values of G form a smooth boundary. Starting with a point in the region to the left of the boundary, i.e. a starting value of G at a given ω, the body will initially slip but will eventually stop with respect to the plate. On the other hand, with a starting point in the

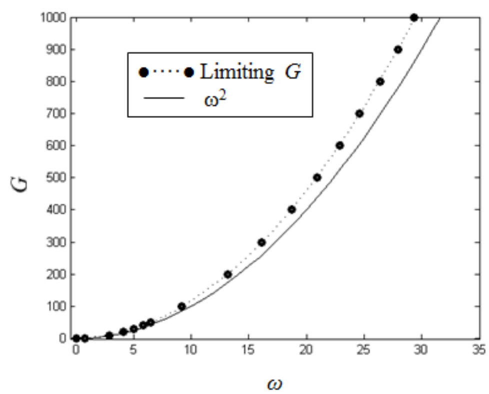

Fig. 3 Limiting G versus the angular velocity of the rotating plate

region to the right of the boundary, the body will eventually move out of the plate.

Also shown in Fig. 3 is a curve representing the relationship between G_1 and ω_{cr} in Eq. (5). As can be seen, the boundary of limiting G is well to the left of the critical angular velocity. This means if a body is place on a rotating plate, it can never reach the position of critical angular velocity as defined in Eq. (5).

4. Conclusion

The problem of a body placed on a plate rotating with a constant angular velocity is investigated. It is found that the body may or may not stop with respect to the plate depending on the relative value of the angular velocity of the plate ω to the parameter G, which depends on the coefficient of friction, the gravitational constant and the initial distance from the center as defined in Eq. (12). The two regions are well separated by a smooth curve which is determined numerically. As expected, the result implies that at a given angular velocity of the rotating plate, a high value of G, which implies high coefficient of friction or small initial radial distance, will ensure that the body would stop with respect to the plate. Conversely, a low value of G resulting from a smooth surface or large initial radial distance would tend to propel the body out of the plate. Attempts to verify the results experimentally will be done in the near future.

7. References

[1] Beer, F.P., Johnston, E.R., Jr., Cornwell, P.J., Self, B., and Sanghi, S. (2016). Vector Mechanics for Engineers: Dynamics, 11th edition, McGraw-Hill Education, Singapore.
[2] Meriam, J.L. and Kraige, L.G. (2011). Engineering Mechanics: Dynamics, 7th edition,John Wiley & Sons, Singapore.
[3] Vongsarnpigoon, L. and Sratong-on, P. (2010) Rotating Horizontal Plate with a Non-Slipping Mass on Top, paper (in Thai) presented at 24th ME-NETT, Ubonrachathani, Thailand.
[4] Simmons, G.F. (1991). Differential Equations with Applications and Historical Notes, 2nd edition, McGraw-Hill, Singapore.