# Two-Wheeled Urban Transportation Vehicle Modeling and Control

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***Abstract***

The dynamics of Two-Wheeled Urban Transportation Vehicle is unstable and highly nonlinear. Like a two-wheeled inverted pendulum robot, the motion of Two-Wheeled Urban Transportation Vehicle is under a nonholonomic constraint which restricts the vehicle to move on its side way. This vehicle has 3-DOF. There are two motors power the left and right wheels. This indicates that it is in the class of under-actuated mechanical system which is a system that has fewer inputs than its degree of freedom. This makes Two-Wheeled Urban Transportation vehicle difficult to be controlled as desired. Thus, it is attractive to researchers worldwide. This paper studies its modeling and control algorithm to balance the vehicle while steering it. A designed PD controller is proposed. Its performance is examined based on the simulation and experiment. The simulation and experiment results are compared and discussed.

***Keywords**:* Control, Urban Transportation Vehicle, Two-Wheeled Inverted Pendulum, Kane's Method

## 1. Introduction

Two-wheeled inverted pendulum has evoked a number of researchers worldwide due to its unique physical [1-4,7]. Its motion is under a nonholonomic constraint which restricts the robot to move on its side way. This robot has 3-DOF. There are two motors powering the left and right wheels. This indicates that it is in the class of under-actuated mechanical system which is a system that has fewer inputs than its degree of freedom. This makes Two-Wheeled Inverted Pendulum robot difficult to be controlled as desired. In this research the robot is equipped with a gyro sensor, acceleration sensor and rotary encoders. They are used to feedback the desired signal used by the controllers. The control algorithm is shown that it has ability to balance and steer the vehicle.

## 2. Dynamic Modeling of Two-Wheeled Urban Transportation Vehicle

The dynamics of two-wheeled urban transportation vehicle is formulated by Kane's method. As illustrated in Fig. 1, the configuration of the vehicle is specified by 3 generalized coordinates ($q_1$, $q_2$, $q_3$). The system has 3 DOF. The system consists of three rigid bodies, two wheels and vehicle body. The joints at each wheel attached to two motors which are rigidly connected to the vehicle body. Thus, they reduce 10 degree of freedom. There are 2 constraints which confine the wheels on the ground. Thus, left and right wheels make the vechile maneuver on the plane. The wheels rotate on the ground without slip (pure rolling). There are two nonholonomic constraints which restrict the wheel from motion toward the slide way. Thus, the robot turns due to different speed between the left and right wheels. The generalize forces are the torque applied by two motors place on the vehicle body. Another inertia force is the gravitation on the vehicle.

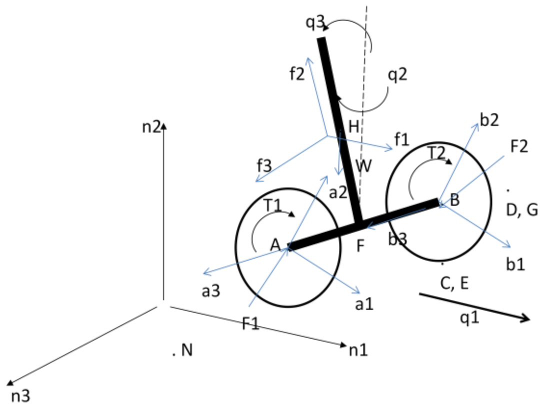


Fig. 1 Free Body Diagram of the vehicle

### 2.1 Derivation of Equation of Motion

Kane's method proposed an effective method to derive a dynamics equation of motion of multi-body system based on partial velocities [2,3]. The required differentiating to compute velocities and accelerations can be obtained through the use of algorithms based on vector products. The constraint can be imposed in the system. Thus, the complexity in derivation of equation of motion is reduced significantly. The equation of motion of two-wheeled urban transportation vehicle can be found and shown in a short form as expressed in Eq. (1).

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + W(q) = Q_u \quad (1)$$

The designed system was assumed to be a nonholonomic system in which no slip occurs between the wheels and the ground. Suppose the configuration of a system is specified by the n generalized coordinates q1, q2,...,qn. Assuming that there are m independent equations of constraint which are written as nonintegerable differential equations. n and m are positive integer number. Constraints of this type are known as nonholonomic constraints. The nonholonomic constraint which explains why the wheels do not slip on the ground.

The generalized coordinates, $q_j$ , are $_1$, $_2$, $q_3$ as shown in Figure 1. $q_1$ is defined as a translational displacement in $\hat{n}_1$ direction. $q_2$ is defined as a angular displacement in $\hat{n}_2$ direction. $q_3$ is defined as a angular displacement in $\hat{n}_3$ direction. Thus, the generalized speeds, $u_i$ , for Kane's method are the first derivative terms of the generalized coordinates determined by the transportation theorem or direct differentiation method. The acceleration terms can be easily obtained by differentiating the generalized speeds. After determining the generalized coordinates, speed and acceleration, then we need to define the forces and torques on the body.

The forces and torque on the vehicle system are assumed to be the forces and torques between the wheels and body of the driving motors, and the gravitational force on the center of gravity of the main body as described in Figure 1. These forces are Fs which are the forces acting on wheels and Ts are torques generated by motors on wheels, W is gravitational force on the body.

Using the velocities, accelerations, angular velocities and angular accelerations already obtained we can derive the generalized active forces and the generalized inertia forces. In the Newtonian reference frame {N}, the generalized active forces, $\tilde{F}_r$ , are constructed. And the generalized inertia forces, $\tilde{F}_r^*$, can be obtained.

Finally, the equations of motion in eq. (4-6) are derived by using the relationship in eq. (3)

$$\tilde{F}_r + \tilde{F}_r^* = 0 \qquad , r = 1,2,3 \quad (3)$$

$$3(M_w + M_b)\ddot{x} - M_b d\cos\emptyset\ddot{\emptyset} + M_b d\sin\emptyset(\ddot{\emptyset}^2 + \dot{\varphi}^2) = -\frac{\alpha_3+\beta_3}{R} \quad (4)$$

$$\left\{\left(3L^2 + \frac{1}{2R^2}\right)M_w + M_b d^2\sin^2\emptyset + I_2\right\}\ddot{\varphi} + M_b d^2\sin\emptyset\, cos\emptyset\, \dot{\emptyset}\dot{\varphi} = \frac{L}{R}(\alpha_3 - \beta_3) \quad (5)$$

$$M_b d\cos\emptyset\ddot{x} + (-M_b d^2 - I_3)\ddot{\emptyset} + M_b d^2 \sin\emptyset\, cos\emptyset\, \dot{\emptyset}^2 + M_b g d sin\emptyset = \alpha_3 + \beta_3 \tag{6}$$

Rearranging the equation of motion in state space representation,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{3(m_w + m_b)(m_b d^2 + I_3) - (m_b d\cos(x_3))^2}\cdot \begin{bmatrix} -m_b d\sin(x_3)(x_4 + x_6)(m_b d^2 + I_3) + \\ (m_b d^2 \sin(x_3)\cos(x_3)x_4^2 + m_b g d\sin(x_3))(m_b d\cos(x_3)) - \\ \frac{\alpha_3 + \beta_3}{R}(m_b d^2 + I_3) - (\alpha_3 + \beta_3)(m_b d\cos(x_3)) \end{bmatrix}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{1}{3(m_w + m_b)(m_b d^2 + I_3) - (m_b d\cos(x_3))^2}\cdot \begin{bmatrix} -m_b^2 d^2 \sin(x_3)\cos(x_3)(x_4^2 + x_6^2) + \\ (m_b d^2 \sin(x_3)\cos(x_3)x_4^2 + m_b g d\sin(x_3))(3(m_w + m_b)) - \\ \frac{\alpha_3 + \beta_3}{R} m_b d\cos(x_3) - (\alpha_3 + \beta_3)(3(m_w + m_b)) \end{bmatrix}$$

$$\dot{x}_5 = x_6$$

$$\dot{x}_6 = \frac{1}{(3L^2 + 0.5R^2)m_w + m_b d^2 \sin^2(\phi) + I_2}\cdot \left[ -m_b d^2 \sin(\phi)\cos(\phi)x_4 x_6 + \frac{L}{R}(\alpha_3 - \beta_3) \right] \tag{7}$$

where

$x_1 = x$, $x_2 = \dot{x}$ , $x_3 = \phi$ , $x_4 = \dot{\phi}$ , $x_5 = \varphi$ , $x_6 = \dot{\varphi}$

$m_b$ is mass of the structure body of the vehicle

$m_w$ is wheel mass

$I_2$ is moment of inertia of the vehicle about it mass center in $n_2$ – direction

$I_3$ is moment of inertia of the vehicle about it mass center in $n_3$ – direction

d is the distance from the center of the vehicle point F to mass center

L is the half distance between left and right wheel

R is the radius of the wheel

**3. Controller Design**

The vehicle is highly nonlinear. Thus, it is difficult to control [5,6,8]. Chosen PD controller is designed based on the mathematical model of eq. (7). The PD gains are tuned to obtain the desired balancing position. Firstly, the purpose of this study is to attempt to control the pitch angle, maintaining the uprising position ($\emptyset \approx 0$). The control law chosen is $u = -K_P\phi - K_D\dot{\phi}$ . In the equation of motion shown above, the 5th and 6th equations are mainly the steering dynamic of the vehicle. In this study, it is not our major objective. It will not effect the uprising control. Third and forth equations are considered as the equation of motion for uprising position control. Our approach, we choose feedback linearization controller. Thus, the control law can be derived from the 3rd and 4th equations.

Let rewritten them in simple form,

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{1}{A - B^2\cos(x_3)}\cdot \begin{bmatrix} -B^2 \sin(x_3)\cos(x_3)(x_4^2) + CBd\sin(x_3)\cos(x_3)x_4^2 + CBg\sin(x_3) \\ -\left(\frac{B\cos(x_1)}{R} + C\right)u \end{bmatrix} \tag{8}$$

where

$$A = 3(m_w + m_b)(m_b d^2 + I_3)$$

$$B = m_b d$$

$$C = 3(m_w + m_b)$$

$u$ is the applied torque ( $\alpha_3 + \beta_3$ ). Since we interest in balancing uprising position, the right and left wheel have the same amount of torque. We want to design feedback controller $u = u(x_3, x_4)$ .

Let introduce the new variable, $z$,

$$z = \dot{x}_4 = \ddot{\phi} \tag{9}$$

which yields the control law

$$u = \frac{-1}{\left(\frac{B\cos(x_1)}{R}+C\right)} \cdot \left[(A-B^2\cos(x_3))z + (B^2 - CBd)\sin(x_3)\cos(x_3)x_4^2 - CBg\sin(x_3)\right] \quad (10)$$

Choosing a PD controller

$$\begin{aligned} z &= z(x_3, x_4) = z(\phi, \dot{\phi}) \\ &= K_P(\phi_{ref} - \phi) + K_D(\dot{\phi}_{ref} - \dot{\phi}) \end{aligned} \quad (11)$$

This yields the closed-loop system.

$$\ddot{\phi} + K_D\dot{\phi} + K_P\phi = K_P\phi_{ref} + K_D\dot{\phi}_{ref} \quad (12)$$

if $\phi_{ref} = 0°$ and $\dot{\phi}_{ref} = 0$ rad/s

it reduces to

$$\ddot{\phi} + K_D\dot{\phi} + K_P\phi = 0 \quad (13)$$

This implies that state $x_3 = \phi$ and $x_4 = \dot{\phi}$ go to zero asymptotically in finite time.

Thus, the feedback controller, *u*, or computed torque is exact linearization or inverse dynamics.

$$u = \frac{-1}{\left(\frac{B\cos(x_1)}{R}+C\right)} \cdot \begin{bmatrix} (A-B^2\cos(x_3))(K_P(\phi_{ref}-\phi) + K_D(\dot{\phi}_{ref}-\dot{\phi})) + \\ (B^2 - CBd)\sin(x_3)\cos(x_3)x_4^2 - CBg\sin(x_3) \end{bmatrix} \quad (14)$$

or

$$u = \frac{-1}{\left(\frac{B\cos(x_1)}{R}+C\right)} \cdot \begin{bmatrix} (A-B^2\cos(x_3))(-K_P(\phi) - K_D(\dot{\phi})) + \\ (B^2 - CBd)\sin(x_3)\cos(x_3)x_4^2 - CBg\sin(x_3) \end{bmatrix} \quad (15)$$

when $\phi_{ref} = 0°$ and $\dot{\phi}_{ref} = 0$ rad/s.

### 3.1 Stability Analysis

For the chosen control law, the vehicle stability is need to be proved to be asymptotically stable. Lyapunov's method of stability analysis is introduced to verify the stability of the system with the proposed feedback linearization controller. Lyapunov's method is the most common way to determine the stability of the nonlinear system such as this vehicle [5,6,8].

Lyapunov's Theorem

1. V(t) is continuous first partial derivatives
2. V(t) is positive definite
3. V'(t) is locally negative definite

We try the standard Lyapunov function candidate

$$V(x_3, x_4) = \frac{1}{2}(x_3^2 + x_4^2) \quad (16)$$

which is radially unbounded, $V(0,0) = 0$ and $V(x_3, x_4) > 0$, $\forall (x_3, x_4) \neq (0,0)$.

The derivative of V is

$$\dot{V} = \dot{x}_3 x_3 + \dot{x}_4 x_4 \quad (17)$$

$$\dot{V} = x_4 x_3 + x_4 \cdot \frac{1}{A - B^2\cos(x_3)} \cdot \begin{bmatrix} -B^2\sin(x_3)\cos(x_3)(x_4^2) + CBd\sin(x_3)\cos(x_3)x_4^2 + \\ CBg\sin(x_3) - \left(\frac{B\cos(x_1)}{R} + C\right)u \end{bmatrix} \quad (18)$$

Inserting control law, u, from previous section

$$\dot{V} < 0 = x_4 x_3 + x_4(-K_P x_3 - K_D x_4) < 0 \quad (19)$$

From above, the condition that makes $\dot{V} < 0$ is true when $K_P$ and $K_D$ are large enough, since $x_4 x_3$ and $x_4^2$ is always positive in both direction when the vehicle decline back and forth.

For the proposed control law, *u*, $\dot{V}$ results in negative definite. Thus, the vehicle is asymptotically stable.

### 3.2 Gain Tuning

From previous section, proportional and derivative gains have to be large enough that keep the first order derivative of Lyapunov function negative definite. Thus, the gains are tuned by trial and error via simulation in MATLAB/Simulink. The acceptable results justify the value of gains. $K_P$ is about 30-40 larger than $K_D$ for our model.

Ideally a motor can produce as much as torque the system requires. But in reality, the motor has limitation of torque production which varies with motor speed. Some source of imperfection in the vehicle such as sensor error, floor surface, disturbance, error in assembling the vehicle, backlash, hysteresis and noise in microcontroller makes tuning procedure very difficult. Thus, the vehicle is built with some degree of less restriction which gives us some tolerance. To tune the controller gain with embedded in the microcontroller; we use the guide line given by the gains from simulation. For ease of programming, the control law is reduced to $u = -K_P\phi - K_D\dot{\phi}$ for programming. The complicated terms in the equation that are eliminated. This is by assuming them to be small number comparing with the gains. After some tedious trial, $K_P$ is 46 and $K_D$ is 1 for our vehicle.

### 4. Simulation and Experiment Results

The simulation and experimental results show in Fig. 2 and 3. In Figure 2, tilt angle is controlled to obtain it original degree which is $0^{\circ}$. From the simulation, the vehicle goes to the desired position at the origin in 2 sec. From the experiment, the vehicle oscillates while maintain the balanced uprising position. Figure 3 illustrates that the tilt speed from the experiment oscillate about 0 rad/s while the tilt speed from the simulation goes to 0 rad/s in around 2 sec. Comparatively, the simulation and experimental results share some agreement which is the vehicle stay in the bound region. Here is around +/- 2 degree about 0 degree positions.

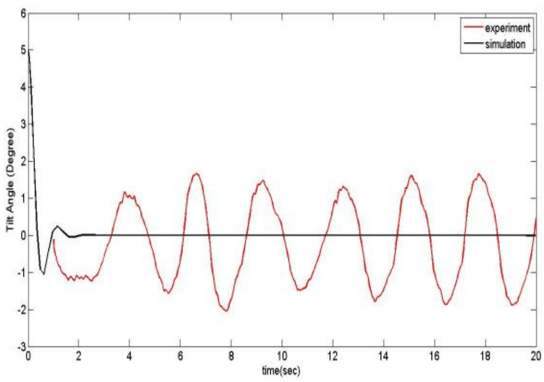


Fig 2. Tilt angle from simulation comparing with experiment

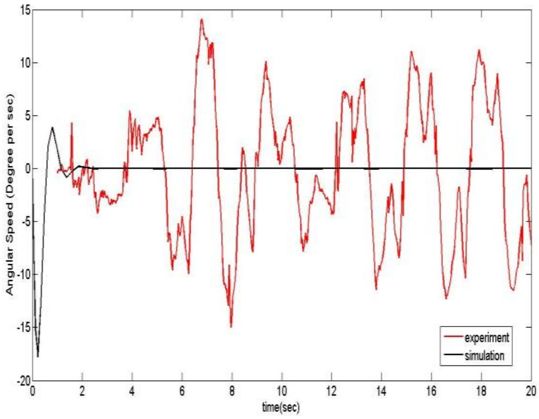


Fig 3. Tilt speed from simulation comparing with experiment

### 5. Conclusion

The equation of motion of the urban transportation vehicle is derived using Kane's Method. Kane's Method shows the systematic approach with ease to use when dealing with the complicated system such as the model in this study. Due to nonlinearity of the system, the controller is designed using feedback linearization. Lyapunov's stability analysis is utilized to ensure the stability of the system with the proposed control law. The modified controller is embedded in to microcontroller to handle the real vehicle. The controllers well perform their capability within the experiment set up after tuning with suggestion from the simulation model. After testing, the vehicle is capable of stay in the uprising position within +/-2 degree. However, in different situation such as incline floor plane or slipping floor, the results might diverse from the result expressed in this research. The refined controller including parameter tuning is required before implementation in other situation.

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