CST0006

Free vibration analysis of functionally graded sandwich beams with elastically constrained ends

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Abstract. In this research, the free vibration of functionally graded (FG) sandwich beams which are supported by translational and rotational springs at both ends is considered by utilizing Chebyshev collocation method. Timoshenko beam theory is employed to construct the governing equations of motion in order to cover the significant effects of shear deformation and rotary inertia. An accuracy of the present modeling is verified by comparing with some existing results in the literature. Moreover, many important parameters such as layer and beam thickness ratios, material volume fraction index, spring constants, etc. are taken into account. Based on numerical results, it is revealed that the stiffness of the springs has significant effect on natural frequencies of the beams and increasing the stiffness leads to the considerable increase of the frequencies.

1. Introduction

A FG sandwich beam is typically composed of multi-layers of high-strength face sheets made of functionally graded materials (FGMs) and flexible core is homogenous material. Due to excellent properties in high strength-to-weight ratio, the use of FG sandwich beams has grown rapidly in various engineering applications such as automotive, marine and aerospace industries. Another advantage of FG sandwich beams is their material properties being changed gradually across the interfaces. Hence, the problems of de-bonding and delaminating modes of failure between layers are eliminated.

In order to understand mechanical behavior of the beams under the action of different loadings, there exist some investigations on static and dynamic responses of the beams in the past few years. Vo et al. [1] investigated vibration and buckling behavior of FG sandwich beams using finite element method (FEM). The relationship between fundamental natural frequency and critical buckling load of the beams was presented. By using higher-order shear deformation theory, Nguyen et al. [2] also provided the solutions for vibration and buckling of FG sandwich beams. A quasi-3D theory was employed to deal with static bending, buckling and vibration problems of FG sandwich beams in Refs. [3-4]. However, all of above studies considered only the beams with general boundary conditions. The investigation on the beams with non-classical boundary conditions is very rare recently. Tossapanon and Wattanasakulpong [5] showed the numerical results of bucking and vibration of FG sandwich beams resting on elastic foundation and the beams were assumed to be supported by classical and non-classical boundary conditions. Trinh et al. [6] also presented the frequency results of FG sandwich beams supported by combinations of non-classical boundary conditions using the state space approach. In terms of dynamic analysis, Bui et al. [7] applied a truly meshfree radial point interpolation method to solve forced vibration of FG sandwich beams under harmonic, heaviside step and transient loadings. For FG sandwich plates and shells, there are some useful investigations [8-14] that can be used as benchmarks for further comparison and design in the field of FG sandwich structures.

In this research, the powerful technique of Chebyshev collocation method that can give very accurate results is adopted to solve the vibration problem of FG sandwich beams with elastically constrained ends. By implementing the method, the present solutions can satisfy all of essential and natural boundary conditions. Our modeling is also useful for designing the beams with imperfect or damaged boundary conditions simulated by using translational and rotational springs. Moreover, some parameters such as material volume fraction index, beam and layer thickness ratios, spring constants, etc., are taken into consideration.

2. Functionally graded sandwich beam

Consider a FG sandwich beam composing of three layers of FG face sheets and homogenous core which is made of ceramic (hardcore) or metal (softcore) as shown in Figure 1. The beam is supported by translational and rotational spring. It is noted that k_{TL}, k_{TR}, k_{RL} and k_{RR} are spring constants of translational and rotational springs at the ends of the beam. The layer thickness ratio of the beam from the bottom $(z = h_0 = -h/2)$ to the top $(z = h_3 = +h/2)$ is defined by three numeric notations.

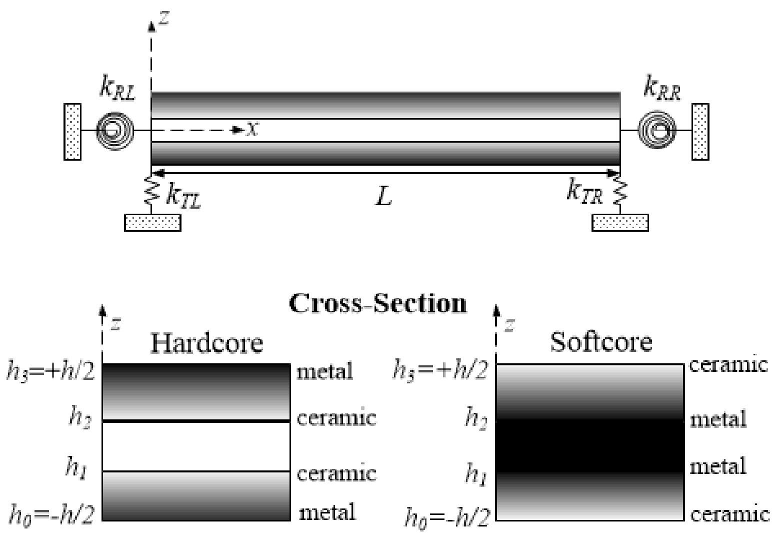

Figure 1. A geometry of FG sandwich beam with elastically constrained ends

The equations for estimating the effective material properties of the beam are given as follows:

$$E^{(i)}(z) = (E_b - E_t)V_b^{(i)}(z) + E_t, \tag{1}$$

$$\rho^{(i)}(z) = (\rho_b - \rho_t)V_b^{(i)}(z) + \rho_t, \tag{2}$$

where $E^{(i)}(z)$ and $\rho^{(i)}(z)$ are the Young's modulus and material density in each layer. The subscripts t and b denote the material properties at the faces and at the core, respectively. The

Poisson's ratio (ν) is assumed to be constant. The material volume fraction, $V_b^{(j)}$, which is based on the power law distribution can be obtained from Ref. [1] as:

$$\begin{cases} V_b^{(1)}(z) = \left(\dfrac{z-h_0}{h_1-h_0}\right)^k & z \in [h_0, h_1] \\ V_b^{(2)}(z) = 1 & z \in [h_1, h_2] \\ V_b^{(3)}(z) = \left(\dfrac{z-h_3}{h_2-h_3}\right)^k & z \in [h_2, h_3] \end{cases} \tag{3}$$

where k is the material volume fraction index or power law index, $0 \le k \le \infty$. The conditions of $(E_b = E_c,\ \rho_b = \rho_c)$ and $(E_t = E_m,\ \rho_t = \rho_m)$ are used for the beam with hardcore; and the beam with softcore, we use $(E_b = E_m,\ \rho_b = \rho_m)$ and $(E_t = E_c,\ \rho_t = \rho_c)$. It is also noted that the subscripts c and m denote the material properties of ceramic and metal phases, respectively.

3. Equations of motion

Based on Timoshenko beam theory, the equations of motion governing vibration behaviour of FG sandwich FG beams can be established as follows:

$$\frac{\partial N_x}{\partial x} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \psi}{\partial t^2} \tag{4a}$$

$$\frac{\partial Q_x}{\partial x} = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{4b}$$

$$\frac{\partial M_x}{\partial x} - Q_x = I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \psi}{\partial t^2} \tag{4c}$$

where u_0 and w_0 are the displacements of a point at the middle plane $(z = 0)$, ψ is the rotation of the beam cross-section and t is time. The mass moment of inertias appearing in Eq. (4) is defined as

$$[I_0, I_1, I_2] = \sum_{i=1}^{3} \int_{h_{i-1}}^{h_i} \rho^{(i)} [1, z, z^2] dz. \tag{7}$$

Additionally, the stress results (N_{xx}, Q_{xz}, M_{xx}) in Eq. (4) can be written in function of material stiffness components $(A_{11},\ A_{55},\ B_{11},\ D_{11})$ as

$$N_x = A_{11} \frac{\partial u_0}{\partial x} + B_{11} \frac{\partial \psi}{\partial x} \tag{5}$$

$$M_x = B_{11} \frac{\partial u_0}{\partial x} + D_{11} \frac{\partial \psi}{\partial x} \tag{6}$$

$$Q_x = K_s A_{55} \left(\frac{\partial w_0}{\partial x} + \psi \right) \tag{7}$$

in which

$$[A_{11},\ B_{11},\ D_{11}] = \sum_{i=1}^{3} \int_{h_{i-1}}^{h_i} E^{(i)} [1,\ z,\ z^2] dz \text{ and } A_{55} = \sum_{i=1}^{3} \int_{h_{i-1}}^{h_i} \frac{E^{(i)}}{2(1+\nu)} dz. \tag{8}$$

To solve the equations of motion in Eq. (4), we have adopted the Chebyshev collocation method (CCM) [15-18] to find out their solutions. Before applying the method, all components in Eq. (4) are required to change into dimensionless form with the Chebyshev domain $(-1 \leq \zeta \geq 1)$. For detail of how to create the dimensionless form of the equations of motion, the readers are referred to the previous study of Tossapanon and Wattanasakulpong [5]. Hence, the new form of Eq. 4 can be rewritten as:

$$4a_{11}\frac{\partial^2 U}{\partial \zeta^2} + 4b_{11}\frac{\partial^2 \varphi}{\partial \zeta^2} = -\bar{I}_0\omega^2 U - \bar{I}_1\omega^2\varphi, \tag{9a}$$

$$K_s a_{55}\left(4\frac{\partial^2 W}{\partial \zeta^2} + 2\eta\frac{\partial \varphi}{\partial \zeta}\right) = -\bar{I}_0\omega^2 W, \tag{9b}$$

$$4\bar{b}_{11}\frac{\partial^2 U}{\partial \zeta^2} + 4\bar{d}_{11}\frac{\partial^2 \varphi}{\partial \zeta^2} - K_s\bar{a}_{55}\eta\left(2\frac{\partial W}{\partial \zeta} + \eta\varphi\right) = -\Phi(\bar{I}_1\omega^2 U + \bar{I}_2\omega^2\varphi). \tag{9c}$$

The left-hand side Eq. (9) can be expressed according to the requirement of the method, using the Chebyshev differentiation matrix as:

$$EM1 = 4a_{11}([1\ 0\ 0]\otimes D_2) + 4b_{11}([0\ 0\ 1]\otimes D_2) \tag{10a}$$

$$EM2 = K_s a_{55}(4([0\ 1\ 0]\otimes D_2) + 2\eta([0\ 0\ 1]\otimes D_1)) \tag{10b}$$

$$\begin{aligned} EM3 = {} & 4\bar{b}_{11}([1\ 0\ 0]\otimes D_2) + 4\bar{d}_{11}([0\ 0\ 1]\otimes D_2)) \\ & - K_s\eta\bar{a}_{55}(2([0\ 1\ 0]\otimes D_1) + \eta([0\ 0\ 1]\otimes I)) \end{aligned} \tag{10c}$$

where $\otimes$ represents the Kronecker product. The size of $EM1, EM2$ and $EM3$ is $N+1\times 3(N+1)$, where N is the number of Chebyshev point. Theses matrices are stacked together in order to produce the $3(N+1)\times 3(N+1)$ global matrix as:

$$EM = \begin{bmatrix} EM1 \\ EM2 \\ EM3 \end{bmatrix}[\delta]^T \tag{11}$$

where $[\delta]^T$ is the transpose displacement vector which can be expressed as:

$$[\delta]^T = [U_1\, U_2 \ldots U_{N+1}\, W_1\, W_2 \ldots W_{N+1}\, \varphi_1\, \varphi_2 \ldots \varphi_{N+1}]^T. \tag{12}$$

The displacements at both ends of the beams are: $U_1\ W_1\ \varphi_1$ and $U_{N+1}\ W_{N+1}\ \varphi_{N+1}$. Therefore, the displacement vector is rewritten as:

$$[\delta]^T = [U_1\, W_1\, \varphi_1\, U_{N+1}\, W_{N+1}\, \varphi_{N+1}\, U_2\, U_3 \ldots U_N\, W_2\, W_3 \ldots W_N\, \varphi_2\, \varphi_3 \ldots \varphi_N]^T. \tag{13}$$

4. Boundary condition equations

In this investigation, the beam is supported by elastic springs at both ends which can be seen in Figure 1. Therefore, the relationship between shear force, bending moment and spring constants at the beam supports can be expressed as:

$$N_x = 0, Q_x - k_{TL}w_0 = 0, M_x - k_{RL}\psi = 0, \tag{14}$$

$$N_x = 0, Q_x - k_{TR}w_0 = 0, M_x - k_{RR}\psi = 0. \tag{15}$$

Similar to the equations of motion, the boundary condition equations must be transformed into the dimensionless form and then we use the Chebyshev differentiation matrix to produce the boundary condition matrices at both ends as follows:

Boundary condition matrices at left end

$$2a_{11}([1\,0\,0]\otimes[D_1(1,:)])[\delta]^T + 2b_{11}([0\,0\,1]\otimes[D_1(1,:)])[\delta]^T = 0, \tag{16}$$

$$2K_s a_{55}([0\,1\,0]\otimes[D_1(1,:)])[\delta]^T + K_s a_{55}\eta([0\,0\,1]\otimes[1\,0\,0...0])[\delta]^T - \beta_{TL}([0\,1\,0]\otimes[1\,0\,0...0])[\delta]^T = 0, \tag{17}$$

$$2\bar{b}_{11}([1\,0\,0]\otimes[D_1(1,:)])[\delta]^T + 2\bar{d}_{11}([0\,0\,1]\otimes[D_1(1,:)])[\delta]^T + \beta_{RL}([0\,0\,1]\otimes[1\,0\,0...0])[\delta]^T = 0, \tag{18}$$

Boundary condition matrices at right end

$$2a_{11}([1\,0\,0]\otimes[D_1(N+1,:)])[\delta]^T + 2b_{11}([0\,0\,1]\otimes[D_1(N+1,:)])[\delta]^T = 0, \tag{19}$$

$$2K_s a_{55}([0\,1\,0]\otimes[D_1(N+1,:)])[\delta]^T + K_s a_{55}\eta([0\,0\,1]\otimes[0\,0\,0...1])[\delta]^T - \beta_{TL}([0\,1\,0]\otimes[0\,0\,0...1])[\delta]^T = 0, \tag{20}$$

$$2\bar{b}_{11}([1\,0\,0]\otimes[D_1(N+1,:)])[\delta]^T + 2\bar{d}_{11}([0\,0\,1]\otimes[D_1(N+1,:)])[\delta]^T + \beta_{RL}([0\,0\,1]\otimes[0\,0\,0...1])[\delta]^T = 0. \tag{21}$$

It is defined that $\beta_{TL} = \dfrac{k_{TL}L}{A_{110}}$, $\beta_{TR} = \dfrac{k_{TR}L}{A_{110}}$, $\beta_{RL} = \dfrac{k_{RL}L}{D_{110}}$ and $\beta_{RR} = \dfrac{k_{RR}L}{D_{110}}$ are dimensionless spring constant factors and A_{110} and D_{110} are A_{11} and D_{11} of homogenous beam. After applying the boundary conditions of beams, the system of algebraic can be expressed as:

$$\begin{bmatrix}[S_{bb}] & [S_{bd}]\\ [S_{db}] & [S_{dd}]\end{bmatrix}\begin{Bmatrix}[\delta_b]\\ [\delta_d]\end{Bmatrix} = \omega^2\begin{bmatrix}[0] & [0]\\ [0] & [M_{dd}]\end{bmatrix}\begin{Bmatrix}[\delta_b]\\ [\delta_d]\end{Bmatrix}. \tag{22}$$

The subscript $'b'$ and $'d'$ refer to the point used for writing the collocation analog of the boundary conditions and the equations of motion, respectively. The size of S_{bb} is 6×6, the size of S_{bd} is $6\times\{3(N+1)-6\}$, the size of S_{db} is $\{3(N+1)-6\}\times 6$ and the size of S_{dd} is $\{3(N+1)-6\}\times\{3(N+1)-6\}$. For another side of Eq. (9), M_{dd} is the dimensionless inertia matrix having the same size as S_{dd}. The M_{dd} can be constructed as follows:

$$[M_{dd}] = -\begin{bmatrix}\bar{I}_0[I] & 0[I] & \bar{I}_1[I]\\ 0[I] & \bar{I}_0[I] & 0[I]\\ \Phi\bar{I}_1[I] & 0[I] & \Phi\bar{I}_2[I]\end{bmatrix} \tag{23}$$

in which $[I]$ is the $(N-1)\times(N-1)$ identiry matrix.

To solve Eq. (22), the first line leads to

$$[\delta_b] = -[S_{bb}]^{-1}[S_{bd}]\{[\delta_d]\}. \tag{24}$$

And the second one yields the following relationship as

$$[S_{db}]\{[\delta_b]\} + [S_{dd}]\{[\delta_d]\} = -\omega^2\{[\delta_d]\}. \tag{25}$$

From above relations, the final algebraic eigenvalue equation can be given by

$$[(-[S_{db}][S_{bb}]^{-1}[S_{bd}] + [S_{dd}]) + \omega^2[M_{dd}]]\{[\delta_d]\} = 0 \tag{26}$$

5. Numerical results and discussion

In this section, we present several numerical exercises for vibration analysis of FG sandwich beams with elastically constrained ends. The FG sandwich beams are made from the mixture of Aluminum (Al) as metal phases and Alumina (Al_2O_3) as ceramic phases. The material properties of the beams such as Young's modulus, mass density and Poisson's ratio are given in Table 1.

Table 1. Material properties of metal (Al) and ceramic (Al_2O_3).

Material	Young's modulus (GPa)	Mass density (kg/m^3)	Poisson's ratio
Aluminum (Al)	70	2702	0.3
Alumina (Al_2O_3)	380	3960	0.3

Convergence study and validation are carried out first in Table 2 to confirm accuracy of our modeling. As can be seen, an accuracy of the present results is improved when number of term (N) increased. For this table, it is defined that Ω is natural frequency in unit of rad/s and $\overline{\omega}$ is dimensionless frequency parameter. The present modeling is adaptable to deal with general boundary conditions. For example, clamped support is obtained when the translational and rotational spring constants being very high $(\approx 10^{12})$ and simply support can be modeled by using zero of translational spring and very high value of rotational spring.

Table 2. Convergence study and validation of dimensionless frequencies of simply supported FG sandwich beams ([1-1-1], homogenous hardcore).

N	$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$
5	4.4691	17.4593	113.5967
7	4.5330	17.9823	39.8887
9	4.5315	17.9415	39.8839
11	4.5316	17.9430	39.7089
13	4.5316	17.9430	39.7182
14	4.5316	17.9430	39.7178
15	4.5316	17.9430	39.7178
Ref.[5]	4.5316	17.9439	39.7226
Ref.[1]	4.5324	-	-

Table 3 presents dimensionless fundamental frequencies of FG sandwich beams with homogenous softcore and the layer thickness ratio of [2-2-1]. The dimensionless frequencies are presented in form of $\omega = \Omega L\sqrt{\frac{I_{00}}{A_{110}}}$ where $A_{110} = E_m h$ and $I_{00} = \rho_m h$. In this table, the length to height ratio (η) and the material volume fraction index (k) are varied. In this exercise, the beams are supported by different general boundary conditions. For this case, increasing η leads to decrease of the frequency; while, the frequency increases as the increase of k

Table 3. Dimensionless fundamental frequencies of FG sandwich beams ([2-2-1]).

B.C.	η	$k=0.5$	$k=1.0$	$k=2.0$	$k=5.0$
S-S	10	0.4394	0.4839	0.5147	0.5338
	15	0.2966	0.3267	0.3475	0.3602
	20	0.2234	0.2461	0.2618	0.2713
C-S	10	0.6657	0.7325	0.7795	0.8092
	15	0.4567	0.5029	0.5350	0.5550
	20	0.3462	0.3813	0.4056	0.4205
C-C	10	0.9316	1.0244	1.0904	1.1334
	15	0.6513	0.7169	0.7628	0.7917
	20	0.4973	0.5476	0.5826	0.6043

In table 4, the frequency parameter (ω) is presented for the case of FG sandwich beams with elastically constrained ends. C-E beam means that the beam is supported by clamped support at left end, while, the right end is supported by elastic springs. The spring constants in this table are set as follows: for C-E beam $(\beta_{TL} = \beta_{RL} = 10^{12}, \beta_{TR} = \beta_{RR} = 10)$, for S-E beam $(\beta_{TL} = 10^{12}, \beta_{RL} = 0, \beta_{TR} = \beta_{RR} = 10)$ and for E-E beam $(\beta_{TL} = \beta_{RL} = \beta_{TR} = \beta_{RR} = 10)$.

Table 4. Dimensionless fundamental frequencies of FG sandwich beams (homogenous hardcore, $\eta = 20, k = 0.5$).

B.C.	FG sandwich beams with symmetrical layers			FG sandwich beams with un-symmetrical layers		
	1-0-1	1-1-1	2-1-2	2-2-1	2-1-1	2-5-3
C-E	0.3851	0.3997	0.3927	0.4056	0.3974	0.4116
S-E	0.2601	0.2693	0.2649	0.2730	0.2679	0.2768
E-E	0.3049	0.3120	0.3085	0.3152	0.3111	0.3183

Figure 2 shows the increase of frequency parameter due to the increase of spring constants for the case of C-E beams $(\beta_{TL} = \beta_{RL} = 10^{12}, \beta_{TR} = \beta_{RR} = 10)$. The frequency of the beam with η=5 is much larger than that of the beams with η=10 and η=15, respectively. The beams with E-E boundary condition $(\beta_{TL} = \beta_{RL} = 10; \beta_{TR} = \beta_{RR} = 10^2)$ are considered in Figure 3 in order to find out the effect of k on frequency results. The variation of k from 0 to 1 shows the dramatic change in frequency. However, for $k > 1$, there is small change in frequency even the value k being increased to 10.

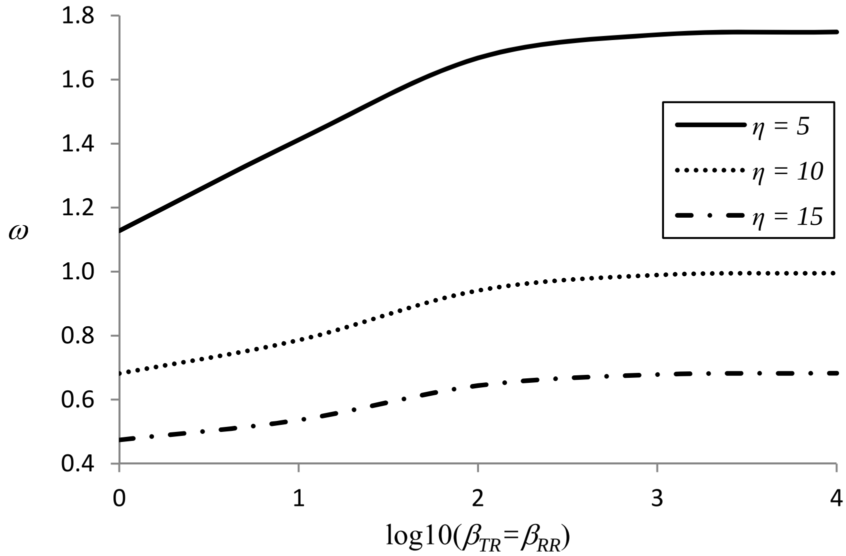

Figure 2. Dimensionless fundamental frequencies of FG sandwich beams with C-E boundary condition (homogenous hardcore, k=0.5, [2-2-1]).

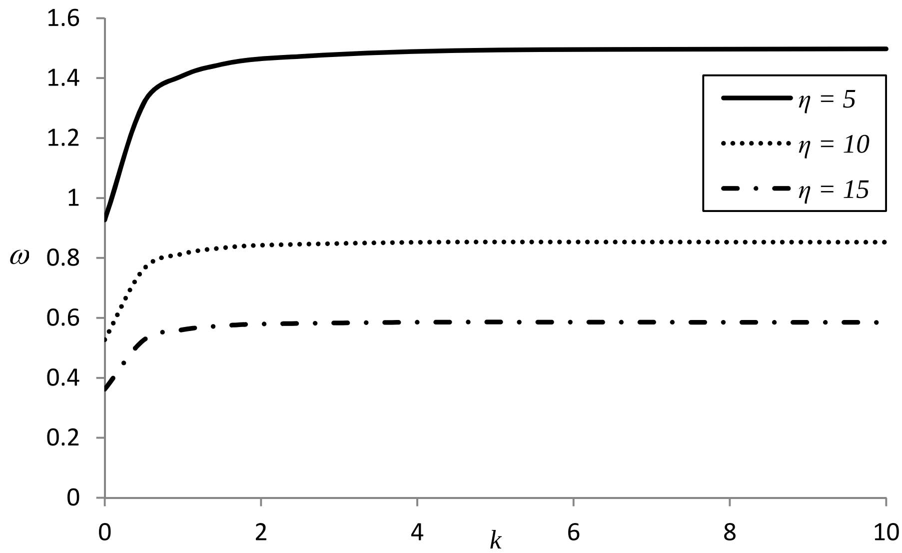

Figure 3. Dimensionless fundamental frequencies of FG sandwich beams with E-E boundary condition (homogenous softcore, [2-1-2]).

Figure 4 illustrates the frequency changes due to the increase of η for three different boundary conditions. The spring constants in this figure are set as follows: for C-E beam $\beta_{TL} = \beta_{RL} = 10^{12}$, $\beta_{TR} = \beta_{RR} = 10$, for S-E beam $\beta_{TL} = 10^{12}$, $\beta_{RL} = 0$, $\beta_{TR} = \beta_{RR} = 10$ and for E-E beam $\beta_{TL} = \beta_{RL} = \beta_{TR} = \beta_{RR} = 10$. As can be observed, the frequency of C-E beam is higher than the frequency of E-E and S-E beams, respectively. To study the significant effect of spring constants on frequency of FG sandwich beams, Figures 5-6 illustrate the variation of fundamental frequency with various values of spring constants. As shown in these figures, it is clearly seen that increasing the constants leads to the considerable increase of frequency for every beam.

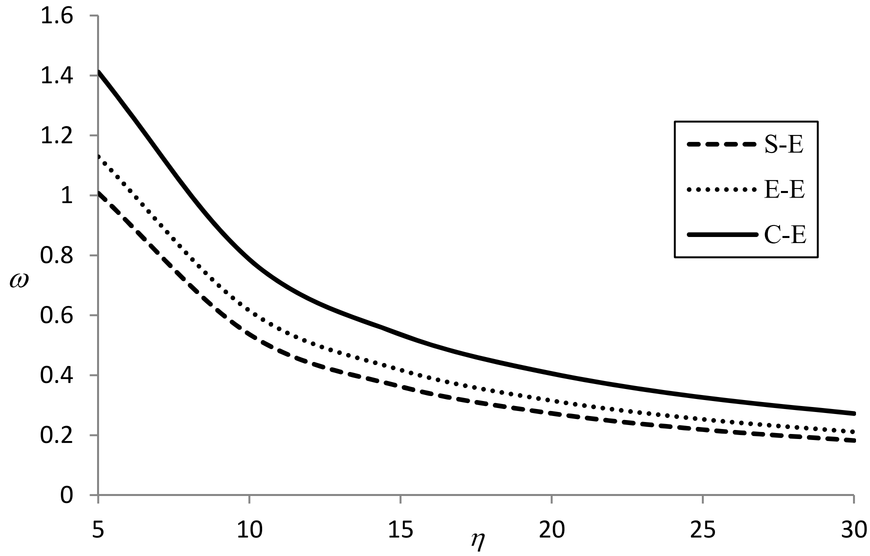

Figure 4. Dimensionless fundamental frequencies of FG sandwich beams with different boundary conditions (homogenous hardcore, [2-2-1], k=0.5).

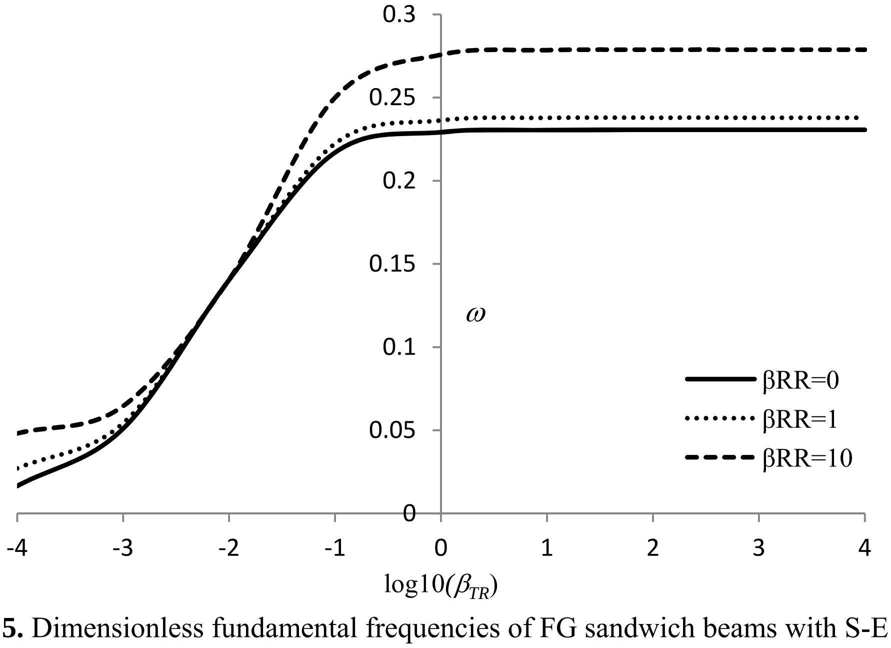

Figure 5. Dimensionless fundamental frequencies of FG sandwich beams with S-E boundary conditions (homogenous softcore, [2-1-1], k=0.5, η=20).

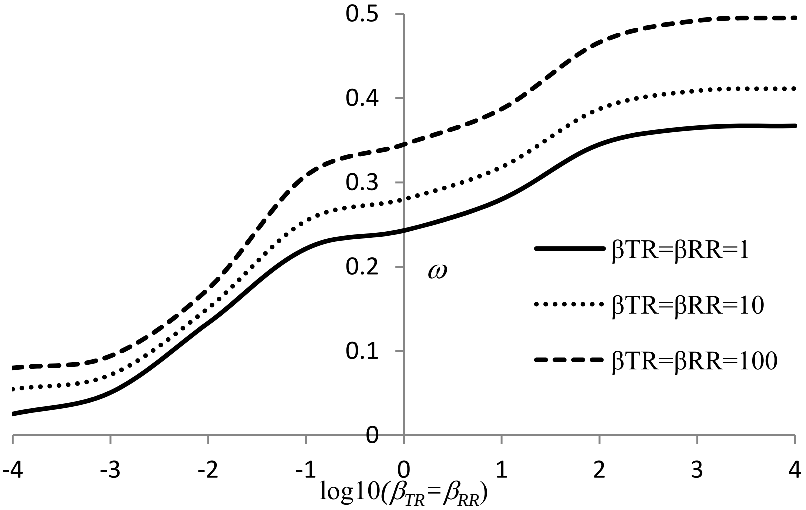

Figure 6. Dimensionless fundamental frequencies of FG sandwich beams with E-E boundary conditions (homogenous softcore, [2-5-3], k=0.5, η=20).

6. Conclusion

In this research, the vibration of FG sandwich beams with elastically constrained ends is analyzed using Chebyshev collocation method. The present solutions satisfy both natural and essential boundary conditions. An accuracy of our modeling is confirmed by comparing with some existing solutions in the past for the cases of general boundary conditions. According to the numerical exercises, it is found that many parameters such as material volume fraction index, layer thickness ratio, length to height ratio, spring constants, etc. have significant impact on the variation of natural frequencies of the beams. For example, the frequency of the thick beam with low value of length to height ratio is larger than the frequency of thin beam with high value of the ratio, for every boundary condition. In case of FG sandwich beams with homogenous softcore, increasing the volume fraction index leads to the increase of the frequency. Due to the system becomes stronger when the spring constants at boundaries increase, therefore, the frequency of the beams is higher as the increase of spring stiffness. As mentioned above, it is important to consider all of those parameters when designing the beams.

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