Formation-Keeping of Uncertain Satellites

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Abstract

A formation-keeping control methodology is proposed that includes both attitude and orbital control requirements in the presence of model uncertainties. The approach develops the requisite control in a two-step process. First, a nominal system model that provides our best assessment of the real-life uncertain system is defined, and a nonlinear controller that satisfies the required attitude and orbital requirements of this nominal system is developed. The controller allows the nonlinear nominal system to exactly track the desired attitude and orbital requirements. Since this closed-form controller assumes that the model of the physical system—the nominal system—has no errors or uncertainties, in the second step an additional additive controller that compensates for model uncertainties is developed. The desired trajectory of the nominal system is used as the tracking signal, and a controller based on a generalization of the concept of a nonlinear damping is developed. The resulting closed-form control causes the desired attitude and orbital requirements of the nominal system to be met in the presence of unknown, but bounded, model uncertainties.

Keywords: Satellite Formation Keeping, Orbital Control, Attitude Control, Model Uncertainties, Nonlinear Damping Control

1. Introduction

The use of small multiple satellites flying in formation holds out the potential for reducing total mission costs, performing certain missions more flexibly and efficiently, and making possible advanced applications such as space interferometry and high resolution imaging [1]. This paper addresses the formation-keeping problem in the presence of model uncertainties. We consider a satellite formation in which a set of follower satellites follows, in a desired manner, a leader satellite. The leader satellite may be a real or fictitious satellite located at a specified location relative to the different follower satellites that constitute the formation. Our aim is to develop a control methodology so that each follower satellite in the formation achieves a desired attitude *and* a desired formation configuration in the presence of uncertainties. The relative trajectories of the follower satellites that comprise the formation (with respect to the leader satellite) may be static in time or they may be required to change dynamically in some prescribed, desired manner.

In the current paper, the control methodology is developed in two steps. The first step uses the concept of the fundamental equation to provide the closed-form control force and torque needed to track the attitude and orbital requirements, for the nominal system model of each satellite. The nominal model is the model adduced from our best assessment of the characteristics of the real-life system. Once the nominal system model is fixed, no linearizations/approximations are made in the description of the dynamics, and the nonlinear controller that exactly satisfies the desired attitude and orbital requirements is obtained. In the next step of the control methodology, this nonlinear controller is augmented by an additional additive controller based on a generalization of the notion of a nonlinear damping. This then provides a general approach to the control of the uncertain satellite system, leading to a set of closed-form nonlinear controllers that satisfy the desired attitude and orbital requirements.

2. The Description of Constrained Mechanical Systems

As stated earlier, we denote the nominal system as our best assessment of the actual real-life system, that is, it is the best deterministic model of the system at hand. It is useful to conceptualize the description of such a nominal multi-body system in a three-step procedure given by the fundamental equation [2]-[10]. Following this procedure, we obtain the explicit equation of motion of the nominal system as

$$M\ddot{q} = Q + Q^c = Q + A^T(AM^{-1}A^T)^+(b - Aa), \quad (2.1)$$

where q is the generalized coordinate n-vector, $M > 0$ is the n by n mass matrix and Q is an n-vector, called the 'given' force, A is a p by n constraint matrix whose rank is r, and b is a p constraint vector, $a = M^{-1}Q$, and the superscript "+" denotes the Moore-Penrose (MP) inverse of a matrix.

The control force that the uncontrolled system is subjected to, because of the presence of the control requirements can be explicitly expressed as

$$Q^c(t) := Q^c(q(t), \dot{q}(t), t) = A^T(AM^{-1}A^T)^+(b - Aa). \quad (2.2)$$

Pre-multiplying both sides of (2.1) by M^{-1}, the acceleration of the nominal system that satisfies the constraints can be expressed as

$$\ddot{q} = a + M^{-1}A^T(AM^{-1}A^T)^+(b - Aa) := a + M^{-1}Q^c(t). \tag{2.3}$$

The generalized control force given in Eq. (2.2) is predicated on our best assessment of the system assuming that this assessment provides an accurate enough deterministic model. Since in real-life situations uncertainties always exist, this control force $Q^c(t)$ needs to be modified to compensate for these uncertainties.

Thus, in order to ensure that the follower satellites, whose models are not exactly known, track the orbital and attitude trajectory requirements of the nominal system, that is, they track the requirements of our best-estimate system, Eq. (2.1) has to be replaced with

$$M_a(q_c, t)\ddot{q}_c = Q_a(q_c, \dot{q}_c, t) + Q^c(t) + Q^u(t), \tag{2.4}$$

where q_c is the generalized coordinate *n*-vector of the controlled actual system and Q^u is the additional control force *n*-vector that compensates for the fact that the model is known only imprecisely, which we shall develop in closed form. The *n* by *n* matrix $M_a := M + \delta M > 0$ is the actual mass matrix of the real-life system which is a function of q_c and *t*, δM is the uncertainty in the mass matrix which may include, among others, say, uncertainties in the masses and moments of inertia of the satellites; the actual 'given' force vector is taken to be $Q_a := Q + \delta Q$ where the *n*-vector Q denotes the nominal 'given' forces, and δQ denotes the *n*-vector of the changes in the 'given' force that are caused by the presence of the uncertainties, such as solar wind. We shall denote the unconstrained acceleration of the actual uncertain system as $a_a := M_a^{-1}Q_a$.

We now refer to Eq. (2.4) as the description of the '*controlled actual system*,' or '*controlled system*,' for short. Pre-multiplying both sides of Eq. (2.4) by M_a^{-1}, the acceleration of the controlled system can be expressed as

$$\ddot{q}_c = a_a + M_a^{-1}Q^c(t) + M_a^{-1}M\,\ddot{u}_c. \tag{2.5}$$

Here $a_a := M_a^{-1}Q_a$ and $Q^u := M\ddot{u}_c$, where $\ddot{u}_c$ is the additional generalized acceleration provided by the additional control forces Q^u to compensate for uncertainties in our knowledge of the actual system and is developed in Section 5.

3. Formation-Keeping Equations of Motion: The Controlled Nominal System

It is assumed that there are N follower satellites and that either a real or a fictitious leader satellite leads this N-satellite formation. The i^{th} follower satellite has a nominal mass $m^{(i)}$ and has a diagonal inertia matrix, $J^{(i)}$, in which the nominal moments of inertia along its body-fixed principal axes of inertia are placed.

It is assumed that the position vector of the center of mass of the i^{th} follower satellite *in the*

Hill frame [11] is given by $\begin{bmatrix} x^{(i)} & y^{(i)} & z^{(i)} \end{bmatrix}^T$ and its orientation is described by the quaternion $u^{(i)} = \begin{bmatrix} u_0^{(i)} & u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \end{bmatrix}^T$. Then we define the generalized displacement 7-vector as

$$q^{(i)}(t) = \begin{bmatrix} x^{(i)} & y^{(i)} & z^{(i)} & u_0^{(i)} & u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \end{bmatrix}^T, \; i = 1, 2, \cdots, N. \tag{3.1}$$

3.1 Uncontrolled Orbital Motion

The inertial orbital motion of the i^{th} follower satellite orbiting the spherical Earth is governed by the relation [13]

$$a_{ECI}^{(i)} = \begin{bmatrix} \ddot{X}^{(i)} \\ \ddot{Y}^{(i)} \\ \ddot{Z}^{(i)} \end{bmatrix} = -\frac{GM_\oplus}{\left(X^{(i)^2} + Y^{(i)^2} + Z^{(i)^2} \right)^{3/2}} \begin{bmatrix} X^{(i)} \\ Y^{(i)} \\ Z^{(i)} \end{bmatrix}, \tag{3.2}$$

where $\begin{bmatrix} X^{(i)} & Y^{(i)} & Z^{(i)} \end{bmatrix}^T$ is the position vector of the center of mass of the i^{th} follower satellite *in the inertial frame* or Earth-centered inertial (ECI) frame [13], G is the universal gravitational constant, and $M_\oplus$ is the mass of the Earth. The corresponding acceleration represented in the Hill frame is given in Ref. [14] as

$$a_{Hill}^{(i)} = -\begin{bmatrix} \ddot{r}_L \\ 0 \\ 0 \end{bmatrix} - 2R\dot{S} \begin{bmatrix} \dot{x}^{(i)} + \dot{r}_L \\ \dot{y}^{(i)} \\ \dot{z}^{(i)} \end{bmatrix} - R\ddot{S} \begin{bmatrix} x^{(i)} + r_L \\ y^{(i)} \\ z^{(i)} \end{bmatrix} - \frac{GM_\oplus}{\left[\left(x^{(i)} + r_L \right)^2 + y^{(i)^2} + z^{(i)^2} \right]^{3/2}} \begin{bmatrix} x^{(i)} + r_L \\ y^{(i)} \\ z^{(i)} \end{bmatrix}. \tag{3.3}$$

Here, $\begin{bmatrix} x^{(i)} & y^{(i)} & z^{(i)} \end{bmatrix}^T$ is the position vector of the center of mass of the i^{th} follower satellite in the Hill frame, r_L is the distance from the center of the Earth to the leader satellite, and R is an orthogonal rotation matrix that maps the ECI frame to the Hill frame, that is,

$$\begin{bmatrix} x^{(i)} + r_L \\ y^{(i)} \\ z^{(i)} \end{bmatrix} = R \begin{bmatrix} X^{(i)} \\ Y^{(i)} \\ Z^{(i)} \end{bmatrix}. \tag{3.4}$$

Each element of the matrix R is given in Ref. [14]. The matrix S in Eq. (3.3) is the active rotation matrix, which is the transpose of R.

3.2 Uncontrolled Rotational Motion

We begin with Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T^{(i)}}{\partial \dot{u}^{(i)}} \right) - \frac{\partial T^{(i)}}{\partial u^{(i)}} = \Gamma_u^{(i)}\left(u^{(i)}, \dot{u}^{(i)}, t \right), \tag{3.5}$$

where $u^{(i)} = \begin{bmatrix} u_0^{(i)} & u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \end{bmatrix}^T$ is the quaternion 4-vector of the i^{th} follower satellite, $\Gamma_u^{(i)}\left(u^{(i)}, \dot{u}^{(i)}, t \right)$ is the 'given' generalized force vector, and $T^{(i)}$ is the rotational kinetic energy of the i^{th} follower satellite which is given by

$$T^{(i)} = \frac{1}{2} \left\{ \omega^{(i)} \right\}^T \hat{J}^{(i)} \left\{ \omega^{(i)} \right\}. \tag{3.6}$$

Here, the 4 by 4 augmented inertia matrix, $\hat{J}^{(i)}$, is defined as

$$\hat{J}^{(i)} := \left[\begin{array}{c|c} J_0^{(i)} & 0 \\ \hline 0 & J^{(i)} \end{array} \right] = \left[\begin{array}{c|ccc} J_0^{(i)} & 0 & 0 & 0 \\ \hline 0 & J_x^{(i)} & 0 & 0 \\ 0 & 0 & J_y^{(i)} & 0 \\ 0 & 0 & 0 & J_z^{(i)} \end{array} \right], \tag{3.7}$$

where $J_0^{(i)}$ is an arbitrary positive number, and $J_x^{(i)}$, $J_y^{(i)}$, $J_z^{(i)}$ are the moments of inertia along the principal axes of the i^{th} follower satellite. Also, the 4 by 1 augmented angular velocity vector, $\{\varpi^{(i)}\}$, is related with quaternions by

$$\{\varpi^{(i)}\} = 2E^{(i)}\dot{u}^{(i)}, \; i = 1, 2, \cdots, N, \qquad (3.8)$$

where $\{\varpi^{(i)}\} = \begin{bmatrix} 0 & \omega_x^{(i)} & \omega_y^{(i)} & \omega_z^{(i)} \end{bmatrix}^T$ and the last three elements, described in the body frame, are the angular velocities about the ECI frame of reference, and the 4 by 4 quaternion matrix $E^{(i)}$ is defined by

$$E^{(i)} := \begin{bmatrix} u^{(i)^T} \\ \hline E_1^{(i)} \end{bmatrix} = \begin{bmatrix} u_0^{(i)} & u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \\ \hline -u_1^{(i)} & u_0^{(i)} & u_3^{(i)} & -u_2^{(i)} \\ -u_2^{(i)} & -u_3^{(i)} & u_0^{(i)} & u_1^{(i)} \\ -u_3^{(i)} & u_2^{(i)} & -u_1^{(i)} & u_0^{(i)} \end{bmatrix}. \qquad (3.9)$$

Substituting Eq. (3.8) into Eq. (3.6) yields the kinetic energy in terms of quaternions

$$T^{(i)} = 2\dot{u}^{(i)^T} E^{(i)^T} \hat{J}^{(i)} E^{(i)} \dot{u}^{(i)}. \qquad (3.10)$$

Then, assuming that there is no applied torque (i.e. $\Gamma_u^{(i)} = 0$), we now apply Lagrange's equation under the assumption that the components of the quaternion 4-vector are all independent of each other, to obtain

$$4E^{(i)^T} \hat{J}^{(i)} E^{(i)} \ddot{u}^{(i)} = -8\dot{E}^{(i)^T} \hat{J}^{(i)} E^{(i)} \dot{u}^{(i)} - 4E^{(i)^T} \hat{J}^{(i)} \dot{E}^{(i)} \dot{u}^{(i)}. \qquad (3.11)$$

This relation can be written in the form $M_u^{(i)} \ddot{u}^{(i)} = Q_u^{(i)}$ by setting $M_u^{(i)} := 4E^{(i)^T} \hat{J}^{(i)} E^{(i)}$, and $Q_u^{(i)}$ to be the right-hand side of Eq. (3.11). It is noted that the mass matrix $M_u^{(i)} = 4E^{(i)^T} \hat{J}^{(i)} E^{(i)}$ is symmetric and positive definite, so it has always its inverse.

It is important to stress that up to now we have assumed that each component of the quaternion vector $u^{(i)}$ is independent of the others. However, to represent a physical rotation of a rigid body we require the quaternion $u^{(i)}$ to have unit Euclidean-norm so that

$$\left\| u^{(i)} \right\|_2^2 = u_0^{(i)^2} + u_1^{(i)^2} + u_2^{(i)^2} + u_3^{(i)^2} = 1. \qquad (3.12)$$

After differentiating twice, we have the following control requirement

$$\begin{bmatrix} u_0^{(i)} & u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \end{bmatrix} \begin{bmatrix} \ddot{u}_0^{(i)} \\ \ddot{u}_1^{(i)} \\ \ddot{u}_2^{(i)} \\ \ddot{u}_3^{(i)} \end{bmatrix} = -\dot{u}_0^{(i)^2} - \dot{u}_1^{(i)^2} - \dot{u}_2^{(i)^2} - \dot{u}_3^{(i)^2} \qquad (3.13)$$

so that [15]

$$\begin{aligned} A_u^{(i)} &= \begin{bmatrix} u_0^{(i)} & u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \end{bmatrix}, \\ b_u^{(i)} &= -\dot{u}_0^{(i)^2} - \dot{u}_1^{(i)^2} - \dot{u}_2^{(i)^2} - \dot{u}_3^{(i)^2} := -N\left(\dot{u}^{(i)}\right). \end{aligned} \qquad (3.14)$$

The resulting rotational equation of motion for the i^{th} follower satellite is thus given by Eq. (2.3)

$$\ddot{u}^{(i)} = -\frac{1}{2} E_1^{(i)^T} J^{(i)^{-1}} \left[\tilde{\varpi}^{(i)}\right] J^{(i)} \varpi^{(i)} - N\left(\dot{u}^{(i)}\right) u^{(i)}, \qquad (3.15)$$

where $E_1^{(i)}$, $J^{(i)}$, and $N\left(\dot{u}^{(i)}\right)$ are defined in Eqs. (3.9), (3.7), and (3.14), respectively, and $\left[\tilde{\varpi}^{(i)}\right]$ is a skew-symmetric matrix defined by

$$\left[\tilde{\omega}^{(i)}\right] := \begin{bmatrix} 0 & -\omega_z^{(i)} & \omega_y^{(i)} \\ \omega_z^{(i)} & 0 & -\omega_x^{(i)} \\ -\omega_y^{(i)} & \omega_x^{(i)} & 0 \end{bmatrix}. \quad (3.16)$$

3.3 Dynamics of Coupled Orbital and Rotational Motion of the Nominal System

In this subsection, we combine the attitude and orbital dynamics. Defining the 7 by 1 generalized displacement vector $q^{(i)}(t)$ as in Eq. (3.1), we have the following equation of uncontrolled motion for each follower satellite from Eqs. (3.3) and (3.15)

$$a^{(i)}(t) = \left(M^{(i)}\right)^{-1} Q^{(i)} = \begin{bmatrix} a_{Hill}^{(i)} \\ \hline -\frac{1}{2} E_1^{(i)^T} J^{(i)^{-1}} \left[\tilde{\omega}^{(i)}\right] J^{(i)} \omega^{(i)} - N\left(\dot{u}^{(i)}\right) u^{(i)} \end{bmatrix}, \quad (3.17)$$

where the 7 by 7 mass matrix is

$$M^{(i)} = \begin{bmatrix} m^{(i)} I_{3\times3} & 0_{3\times4} \\ 0_{4\times3} & M_u^{(i)} \end{bmatrix}, \quad (3.18)$$

and $M_u^{(i)} := 4E^{(i)^T} \hat{J}^{(i)} E^{(i)}$ is previously defined.

When the trajectory/orientation requirements are imposed, the generalized control force required to follow them is explicitly obtained by Eq. (2.2), as shall be shown in the next subsection. In addition, we can relate the 4 by 1 generalized quaternion torque, $\Gamma_u^{(i)}$, which is determined by Eq. (2.2), to the 3 by 1 physically applied torque $\Gamma^{(i)} = \begin{bmatrix} \Gamma_x^{(i)} & \Gamma_y^{(i)} & \Gamma_z^{(i)} \end{bmatrix}^T$, about the body axis of the i^{th} follower satellite, through the relation [15]

$$\begin{bmatrix} 0 \\ \Gamma^{(i)} \end{bmatrix} = \frac{1}{2} E\, \Gamma_u^{(i)},\ i = 1, 2, \cdots, N. \quad (3.19)$$

3.4 Determination of the Control Forces and Torques Using the Fundamental Equation

In this subsection, an explicit form of the generalized control force and torque is obtained via the fundamental equation, assuming no uncertainties in the masses and the moments of inertia of the follower satellites. These forces are obtained based on the description of the nominal system. Also, it is assumed for brevity that there is only one follower satellite in the formation and the leader satellite is in a circular orbit with constant radius of r_L around a uniform spherical Earth. Now we consider the following attitude and orbital requirements: (1) The follower satellite's orbit should be on a circle with constant radius ρ_0 when projected onto the *yz*-plane of the Hill frame with the leader satellite located at the center of the circle (this orbit is called the projected circular orbit (PCO) [16]); and, (2) The follower satellite, more specifically, the *z*-axis of its body frame, points to the center of the Earth at all times. (3) Besides these two requirements, we shall impose the additional constraint that the quaternion 4-vector of the follower satellite should have unit norm. These trajectory requirements are summarized as:

$$2x = z\,,\;\; y = \rho_0 \cos(\omega t)\,,\;\; z = \rho_0 \sin(\omega t), \qquad (3.20a)$$

$$[\tilde{z}_b]\; P\begin{bmatrix} -X \\ -Y \\ -Z \end{bmatrix} = 0, \qquad (3.20b)$$

and

$$N(u) = u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1. \qquad (3.20c)$$

The position vector in the ECI frame in Eq. (3.20b) can be transformed to the one in the Hill frame, and vice versa, by using the relation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R^T \begin{bmatrix} x + r_L \\ y \\ z \end{bmatrix} = \begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \begin{bmatrix} x + r_L \\ y \\ z \end{bmatrix}, \qquad (3.21)$$

where the components of the transformation matrix R are given in Ref. [14]. In Eq. (3.20b), $[\tilde{z}_b]$ is the skew-symmetric matrix given by

$$[\tilde{z}_b] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3.22)$$

corresponding to the unit vector along the *z*-axis of the body frame $\hat{z}_b = [0 \quad 0 \quad 1]^T$, and P in Eq. (3.20b) is a transformation matrix that maps the ECI frame into the body frame of the follower satellite which is of the form [15]

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$
$$= \begin{bmatrix} u_0^2 + u_1^2 - u_2^2 - u_3^2 & 2(u_1 u_2 - u_0 u_3) & 2(u_0 u_2 + u_1 u_3) \\ 2(u_1 u_2 + u_0 u_3) & u_0^2 - u_1^2 + u_2^2 - u_3^2 & 2(u_2 u_3 - u_0 u_1) \\ 2(u_1 u_3 - u_0 u_2) & 2(u_0 u_1 + u_2 u_3) & u_0^2 - u_1^2 - u_2^2 + u_3^2 \end{bmatrix}. \qquad (3.23)$$

Eq. (3.20b) originates from the fact that the desired pointing axis (i.e. *z*-axis of the body frame) is constrained to point along the vector connecting the follower satellite and the center of the Earth in the ECI frame, $[-X \quad -Y \quad -Z]^T$. The components of this vector, in turn, are transformed into the body frame by the transformation matrix P, and the cross product of this transformed vector and the *z*-axis of the body frame is zero because they are parallel.

Then differentiating the constraints 3.20 to the form ([2]-[10])

$$A\ddot{q} := \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \ddot{q} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} := b, \qquad (3.24)$$

the control force and torque for the nominal system to satisfy the given orbital and attitude requirements are explicitly determined by Eq. (2.2).

4. Generalized Nonlinear Damping Controller

Our aim in this section is to develop a compensating controller that can guarantee tracking of the nominal system's trajectory in the presence of uncertainties in the actual satellite system. To do this we use a generalization of the concept of a nonlinear damping [17]. The formulation permits the use of a large class of control laws that can be adapted to the practical limitations of the specific compensating controller being used and the extent to which we want to compensate for the uncertainties.

To obtain the additional controller $\ddot{u}_c$, we first define the tracking error for the satellite system (difference between the state of the controlled actual system and the controlled nominal system),

$$e(t) = q_c(t) - q(t). \qquad (4.1)$$

To ensure the motion of the controlled actual system closely tracks the motion of the nominal system and thereby satisfies the control requirements (2.4), we apply an additional compensating controller from a generalized idea of the nonlinear damping, which is explicitly given as,

$$\ddot{u}_c = -\left(e_1(t) + f(e_2)\right) - ke_2(t), \qquad (4.2)$$

where $e_1 := e,\ e_2 := \dot{e}$, and $k > 0$ is arbitrary small positive constants. The *i*-th component, $f_i(e_2)$, of the *n*-vector $f(e_2)$ is defined as

$$f_i(e_2) = g\left(e_{2,i} / \varepsilon\right),\ i = 1, \ldots, n \qquad (4.3)$$

where $e_{2,i}$ is the *i*-th component of the *n*-vector e_2, ε is defined as any small positive number and the function $g\left(e_{2,i} / \varepsilon\right)$ is *any* arbitrary strictly increasing ,continuous, odd function of $e_{2,i}$ on the interval $(-\infty, +\infty)$ and it goes to ∞ as $e_{2,i}$ goes to ∞.

Main Result: The closed-from generalized damping controller for the uncertain system,

$$\begin{aligned} M_a \ddot{q}_c &= Q_a + Q^c(t) + M\ddot{u}_c \\ &= Q_a + Q^c(t) - M\left[\left(e_1(t) + f(e_2)\right) + ke_2(t)\right], \end{aligned} \qquad (4.4)$$

where:

(i) the control force $Q^c(t)$ is given by (2.2)

$$Q^c(t) = A^T(AM^{-1}A^T)^+(b - Aa) \qquad (4.5)$$

and is obtained on the basis of the nominal system;

(ii) $k > 0$ is arbitrary small positive number; and

(iii) $f(e_2)$ is *any* arbitrary strictly increasing odd continuous function of e_2 on the interval $(-\infty, +\infty)$ and goes to ∞ as e_2 goes to ∞,

will cause the actual system to track the trajectory of the nominal system.

5. Numerical Results and Simulations for Attitude and Orbital Controls

Let us consider a system in which there is only one follower satellite whose nominal mass is $m = 120\ \mathrm{kg}$. Also, its nominal moments of inertia along its respective body-fixed axes are taken to be $J = diag\left(10 \quad 10 \quad 7.2\right)\ \mathrm{kg \cdot m^2}$. The value of J_0 is chosen as $15\ \mathrm{kg \cdot m^2}$ (see Eq. (3.7)). By nominal we mean our best-estimate of these parameters for the actual, real-life, follower satellite. As previously assumed, the leader satellite is in a circular orbit around a uniform spherical Earth and the radius of its orbit is $r_L = 7 \times 10^6\ \mathrm{m}$. For the sake of simplicity, the inclination (i_L) of the leader satellite's orbit is taken to be 0° (that is, the leader satellite is just above the equator), and it is assumed that the leader satellite is on the *X*-axis of the ECI frame at the initial time ($t=0$). The leader satellite's mean motion and orbital period are given by, respectively,

$$n_L = \sqrt{\frac{GM_\oplus}{r_L^3}} = 1.0780\times10^{-3}\ \text{rad/s}, \qquad P_L = \frac{2\pi}{n_L} = 5.8285\times10^{3}\ \text{s} = 1.6190\ \text{hr}. \tag{5.1}$$

We choose $2P_L$ (two orbital periods of the leader satellite) as the duration of time used for numerical integration and the three orbital and attitude requirements are applied to the formation system, which introduced in Section 3. For the radius of PCO in Eq. (3.20a), $\rho_0 = 7.0\times10^4$ m is chosen and the constant rotational frequency ω (see Eq. (3.20a)) is set to equal n_L in Eq. (5.1)

We choose the initial conditions for orbital motion of the follower satellite as

$$x(0)=0\ \text{m},\ y(0)=7.0\times10^4\ \text{m},\ z(0)=0\ \text{m}, \quad \dot{x}(0)=37.7347\ \text{m/s},\ \dot{y}(0)=0\ \text{m/s},\ \dot{z}(0)=75.4695\ \text{m/s}, \tag{5.2}$$

and the initial conditions for attitude motion as

$$u_0(0)=0.0707372,\ u_1(0)=0.997482,\ u_2(0)=0.00498729,\ u_3(0)=3.536772\times10^{-4},\ \dot{u}_0(0)=-0.00870185,\ \dot{u}_1(0)=6.143960\times10^{-4},\ \dot{u}_2(0)=5.403876\times10^{-4},\ \dot{u}_3(0)=0. \tag{5.3}$$

Application of the control force (Eq. (2.2)) yields the trajectories of the motion of the follower.

Figure 1 represents the orbit of the follower satellite projected on the yz-plane (left) and xz-plane (right) in the Hill frame, respectively. The scale is normalized by ρ_0. In Fig. 2, the time history of each component of the quaternion for the follower satellite is shown where time is normalized by P_L. In Fig. 3 we show the obtained control forces per unit mass of the follower satellite in order to follow the desired orbital requirements. The force components are described in the Hill frame, and time is normalized by the period of the leader satellite (i.e. P_L). Figure 4 illustrates the control torques per unit mass of the follower satellite for satisfying the attitude requirements. The torque components are described in the body frame of the follower satellite using Eq. (3.19). Figure 5 represents errors in satisfying the desired nominal trajectories assuming no uncertainties, described by Eq. (3.20). Instead of Eq. (3.20b), we use the angle θ between the *z*-axis of the body frame and the vector $P[-X \quad -Y \quad -Z]^T$ connecting the follower satellite and the center of the Earth. We denote these errors by (a) $e_1(t)=2x-z$, (b) $e_2(t)=y-\rho_0\cos(\omega t)$, (c) $e_3(t)=z-\rho_0\sin(\omega t)$, (d) $e_4(t)=\theta$, and (e) $e_5(t)=u_0^2+u_1^2+u_2^2+u_3^2-1$.

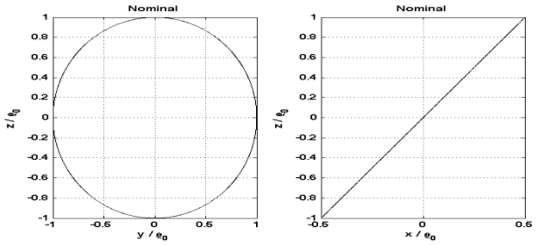

Fig. 1 Constrained motion of the nominal system with no uncertainties assumed

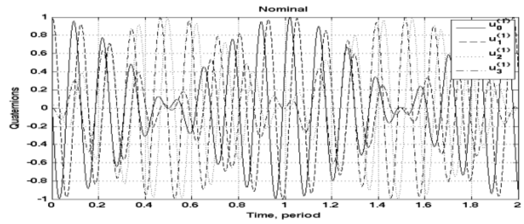

Fig. 2 Time history of quaternions of the nominal system with no uncertainties assumed

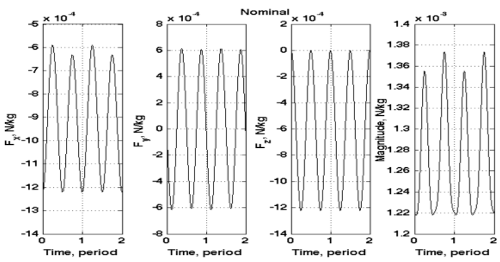

Fig. 3 Required control forces to satisfy the nominal orbital constraints

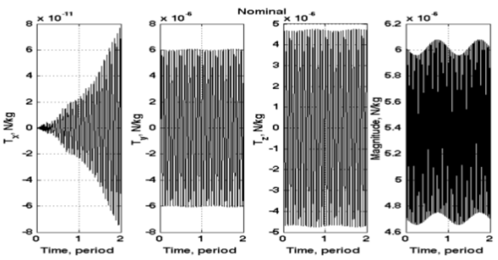

Fig. 4 Required control torques to satisfy the nominal attitude constraints

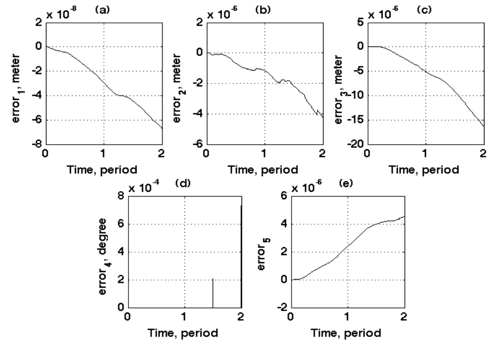

Fig. 5 The errors in the satisfaction with the nominal constraints

To see how the response of the assumed nominal system can be altered through the effect of the uncertainty in the modeling process, we consider for reasons of simplicity only the uncertainties in the mass m of the follower satellite and in its moments of inertia J_x, J_y, and J_z. We estimate that the actual values of these parameters differ from our nominal (best-estimate) values by a random uncertainty of $\pm 10\%$ of the nominal values chosen. For illustrative purposes, we pick a specific system with $\delta m = 12,\ \ \delta J_x = 1,\ \ \delta J_y = 1,$ and $\delta J_z = 0.72$ and perform a simulation again using Eq. (2.3), except that we replace our best-estimate mass matrix of the uncontrolled system with the actual mass matrix $M_a := M + \delta M$, where M is defined in Eq. (3.18), with all other parameter values the same as previously prescribed. We note that the elements of the 7 by 7 symmetric matrix M_a are given in a manner similar to Eq. (3.18). In this case, we have replaced m and J_i in Eq. (3.18) with $m = m + \delta m$ and $J_i = J_i + \delta J_i$ respectively. Using the control forces ($Q^c(t)$), we obtain

$$\ddot{\tilde{q}} := M_a^{-1} Q(\tilde{q}, \dot{\tilde{q}}, t) + M_a^{-1} Q^c(t). \tag{5.4}$$

The trajectories ($\tilde{q}, \dot{\tilde{q}}$) of the system Eq. (5.4) with the actual mass and moments of inertia are determined.

Figure 6 shows these orbital trajectories of the actual system projected on the yz-plane (left) and xz-plane (right) in the Hill frame, respectively. Figure 7 depicts the time history of quaternions of the actual uncertain system. Both figures are different from those obtained from the nominal system, showing that a miss-assessment of the mass and moments of inertia of the follower satellite can have significant consequences. The resulting quaternions satisfy neither the Earth-pointing constraint nor the unit-norm constraint.

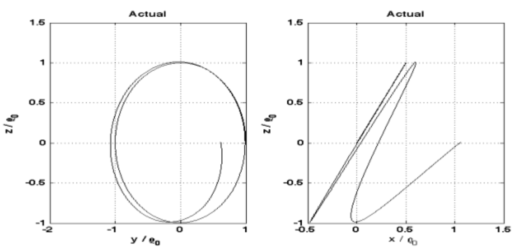

Fig. 6 Motion of the actual system when $\pm 10\%$ uncertainties in the mass and moments of inertia of the follower satellite are involved

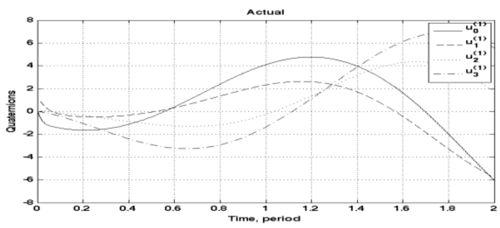

Fig. 7 Quaternions of the actual system when $\pm 10\%$ uncertainties in the mass and moments of inertia of the follower satellite are involved

We next select the structure and parameters for the controller $\ddot{u}_c$. We choose

$$f_i(e_2) = (e_{2,i} / \varepsilon)^3, \tag{5.5}$$

where $\varepsilon > 0$ is a suitable small number, and obtain in closed-form the additional controller needed to compensate for uncertainties in the actual system as

$$\ddot{u}_c(t) = -\left(e_1 + ke_2\right) - (e_2 / \varepsilon)^3 \tag{5.6}$$

Pre-multiplying both sides of Eq. (4.4) by M_a^{-1} and using the additional controller Eq. (5.6), we obtain the closed-form equation of motion of the controlled actual system

$$\ddot{q}_c = a_a + M_a^{-1} Q^c(t) - M_a^{-1} M \left(\left(e_1 + ke_2\right) + (e_2 / \varepsilon)^{1/3}\right) \tag{5.7}$$

which will cause the actual system to track the trajectory of the nominal system, thereby compensating for the uncertainty in our knowledge of the actual system.

For our simulation we choose: $k = 1$ and $\varepsilon = 10^{-2}$. At the scales shown, the controlled trajectories of the follower satellite projected on the yz-plane and xz-plane in the Hill frame fall exactly on those shown in Fig. 1. The errors in tracking the attitude and the orbital trajectories of the nominal system are shown in Fig. 8 and Fig. 9. In Fig. 8, we see that there are minute differences between the position coordinates of the nominal and the controlled trajectories of the follower satellite, and the attitude differences, as represented by the quaternion vectors, are also small as seen in Fig. 9.

Figures 10 and 11 respectively show the additional control forces and torques per unit mass of the follower satellite in order to compensate for the uncertainties in the mass and moments of inertia. Both additional control forces and torques are seen to be small when compared with those obtained from the nominal system (Fig. 3 and Fig. 4, respectively).

We note that use of the specified smooth cubic function $f_i(e_2)$ given in (5.5) eliminates chattering.

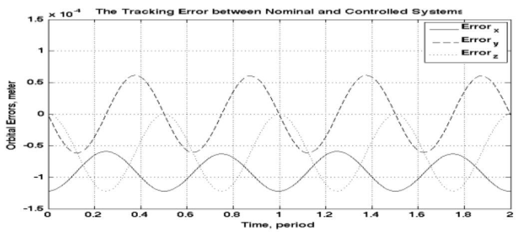

Fig. 8 Orbital errors between the nominal and controlled systems

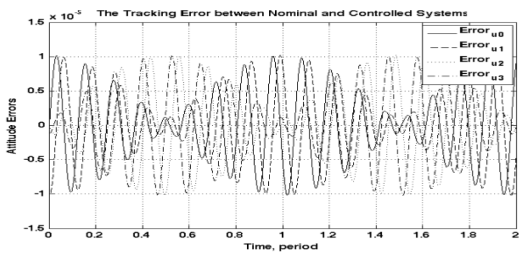

Fig. 9 Attitude (quaternion) errors between the nominal and controlled systems

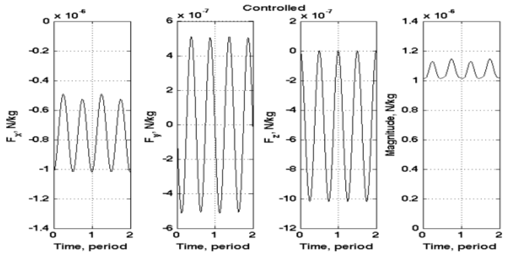

Fig. 10 Required additional control forces

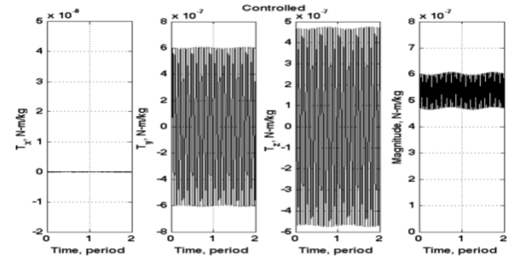

Fig. 11 Required additional control torques

6. Conclusions

In this paper, a simple method for the formation-keeping problem with attitude and orbital requirements, in the presence of model uncertainties has been developed.

The main contributions of the paper are the following:

1. We obtain the exact closed-form solution (for the nominal dynamical system assumed) to the problem of formation-keeping with both attitude and orbital requirements. Unlike previous research, we directly start from the nonlinear equation, and the exact control force and torque are easily obtained, considering all the nonlinearities of a multi-satellite system.

2. Since the control force and torque to be applied to the follower satellites are explicitly

obtained in closed-form and the method is not computationally intensive, it can be easily used for on-orbit, real-time control of maneuvers, especially for formations with many satellites, for which the underlying dynamics is highly nonlinear.

3. A general closed-form controller for keeping a satellite formation so that it satisfies desired attitude and orbital requirements in the presence of model uncertainties has been developed. This is obtained by adding to the nominal controller an additional control that compensates for the uncertainties. Uncertainties in the two dynamical quantities *M* and *Q* that characterize the system can be accommodated.

4. The control function $f_i(e_2)$ and the parameters that define the compensating controller can be chosen depending on a practical consideration of the control environment, and on the extent to which the compensation of uncertainties is desired. Thus when dealing with large, complex multi-body systems greater flexibility is afforded. For example, the use of a cubic function may obviate the need for a high-gain controller and would also allow the continuous control, thereby preventing chattering.

5. For brevity, we have illustrated through numerical examples uncertainties that are related to the mass and moments of inertia of the follower satellite. However, the formulation of the current methodology encompasses both general sources of uncertainties—uncertainties in the description of the physical system and uncertainties in knowledge of the 'given' force applied to the system. The closed-form controller developed herein is therefore general enough to be applicable to complex dynamical system of multi-satellites in which the uncertainties in the given force may be important.

7. References

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