Nonlinear Damping Control for Uncertain Nonlinear Multi-body Mechanical Systems

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Abstract

Descriptions of real-life complex multi-body mechanical systems are usually uncertain, and their effective control must take into account uncertainties that arise from two general sources: uncertainties in the knowledge of the physical system and uncertainties in the 'given' forces applied to the system. Both categories of uncertainties, which we assume to be time varying and unknown, yet bounded, are considered in this paper. In the face of such uncertainties, what is available in hand is therefore just the so-called 'nominal system,' which is simply our best assessment and description of the actual real-life situation. The aim of this paper is to develop a general control methodology, which when applied to a real-life uncertain multi-body system, causes this system to track a desired reference trajectory that is pre-specified for the nominal system to follow. An example of a simple mechanical system demonstrates the efficacy and ease of implementation of the tracking control methodology.

Keyword: Constrained Motion, Energy Control, Uncertain System, Nonlinear Damping Control

1. Introduction

All real-life physical systems are known only to within some bounds of uncertainty that may depend on the various levels of their description. Controlling the motion of such uncertain complex multi-body systems to follow prescribed reference trajectories has become a topic of great interest during the past few years. The uncertainties that arise in complex mechanical systems stem from two main sources: (i) uncertainties in our knowledge of the physical system, like uncertainties in the stiffness and mass distribution, the nature of damping, etc.; and, (ii) uncertainties in our knowledge of the externally applied forces acting on the system, like air drag, solar wind, gravity gradients, etc., when considering, for example, precise satellite motion control. The two sources of uncertainty are simultaneously considered in this paper, and in what follows, all these uncertainties are included in what we call the 'real-life mechanical system,' or the '*actual system*,' whose description is known only imprecisely. While not known precisely, it is assumed, however, that these uncertainties are, in general, time varying and unknown, but bounded. Our best assessment of a given actual system will be referred to as the '*nominal multi-body system*,' or the '*nominal system*,' for short. This term naturally includes the best assessment of our characterization of the physical system and of the nature of the 'given' forces acting on it.

The aim of this paper is to develop a general control methodology for determining the control, which when applied to an 'actual system,' causes this system to follow the trajectory that is pre-specified by the control requirements imposed on the corresponding 'nominal system,' and thereby to satisfy the control requirements of the nominal system. The control methodology is developed in a two-step process. The first step uses the concept of the fundamental equation to provide the closed-form control force needed to satisfy the control requirements imposed on the nominal system model. Upon specification of the nominal system model, no linearizations/approximations are made in the description of its dynamics, and the nonlinear controller that exactly satisfies the desired control requirements is obtained in closed form [1-3]. In the next step of the control methodology, this nonlinear controller is augmented by an additional additive controller based on a generalization of the notion of a nonlinear damping. This then provides a general approach to the control of nonlinear uncertain systems, leading to closed-form nonlinear controllers that can guarantee satisfaction of the prescribed control requirements.

2. On the Dynamics of the Nominal Multi-body Systems

2.1 System description of the nominal system

We begin by introducing the description of the nominal system, by which we mean our best assessment of the 'actual system,' whose description is known only imprecisely. It is useful to conceptualize the description of such a nominal multi-body system in a three-step procedure [4-6]. We do this in the following way:

First, we describe the uncontrolled system in which the coordinates are all assumed independent of each other. The equation of motion of this system is given, using Lagrange's equation, by

$$M(q,t)\ddot{q} = Q(q,\dot{q},t), \qquad (2.1)$$

with the initial conditions

$$q(t=0) = q_0,\ \dot{q}(t=0) = \dot{q}_0, \qquad (2.2)$$

where *q* is the generalized coordinate *n*-vector, $M > 0$ is the *n* by *n* mass matrix which is a function of *q* and *t*, and *Q* is an *n*-vector, called the 'given' force, which is a known function of q, $\dot{q}$, and *t*.

From Eq. (2.1) we find the acceleration of the uncontrolled system given by

$$a := M^{-1}(q,t)\, Q(q,\dot{q},t). \qquad (2.3)$$

Second, we impose a set of control requirements as constraints on this uncontrolled system. We suppose that the uncontrolled system is now subjected to the *m* sufficiently smooth control requirements given by [6]

$$\varphi_i(q,\dot{q},t) = 0, \quad i = 1,2,...,m, \qquad (2.4)$$

where $r \le m$ equations in the equation set (2.4) are functionally independent. The control constraints described by (2.4) include all the usual varieties of holonomicand/or nonholonomic constraints. The presence of the control requirements does not permit all the components of the *n*-vectors q_0 and $\dot{q}_0$ to be independently assigned. We shall assume that the initial conditions (2.2) satisfy the *m* control requirements. (If not, the control constraints can be expressed in an alternative form so that they are asymptotically satisfied [7], see Section 2.2.)

Differentiating the control requirements (2.4) with respect to time *t* we obtain the relation [8]

$$A(q,\dot{q},t)\,\ddot{q} = b(q,\dot{q},t), \qquad (2.5)$$

where *A* is an *m* by *n* matrix whose rank is *r*, and *b* is an *m*-vector. We note that each row of *A* arises by appropriately differentiating one of the *m* control requirements in the set given in (2.4).

In the third step, the equation of motion of the '*controlled nominal system*,' or the '*nominal system*' is given by

$$M(q,t)\,\ddot{q} = Q(q,\dot{q},t) + Q^c(q,\dot{q},t), \qquad (2.6)$$

where Q^c is the control force *n*-vector that arises to ensure that the control requirements (2.5) are satisfied. The explicit equation of motion of the nominal system is given by the *fundamental equation* [3, 7]

$$M\ddot{q} = Q + A^T(AM^{-1}A^T)^+(b - Aa), \qquad (2.7)$$

wherein the various quantities have been defined in the previous two steps and the superscript "+" denotes the Moore-Penrose (MP) inverse of a matrix. In the above equation, and in what follows, we shall suppress the arguments of the various quantities unless required for clarity.

The control force that the uncontrolled system is subjected to, because of the presence of the control requirements (2.4), can be explicitly expressed as

$$Q^c(t) := Q^c(q(t),\dot{q}(t),t) = A^T(AM^{-1}A^T)^+(b - Aa). \qquad (2.8)$$

The control force given in (2.8) is optimal in the sense that it minimizes the control cost $Q^{cT} M^{-1} Q^c$ at *each* instant of time [7, 8].

Pre-multiplying both sides of (2.7) with M^{-1}, the acceleration of the nominal system that satisfies the constraint (2.4) can be expressed as

$$\ddot{q} = a + M^{-1} A^T (AM^{-1}A^T)^+ (b - Aa) := a + M^{-1} Q^c(t). \tag{2.9}$$

2.2 Example

To demonstrate the applicability of the control methodology, we introduce an example of a simple multi-body system. We will continue this example all the way through this paper. It is straightforward to extend this example to more general situations.

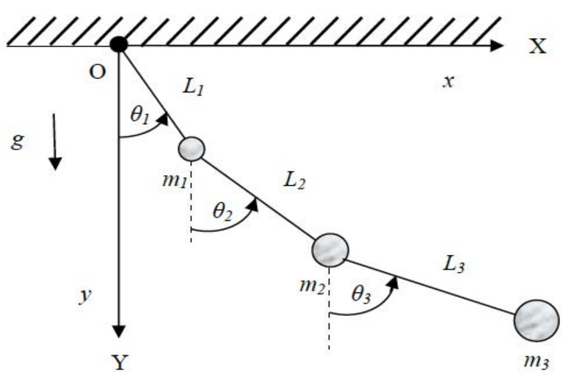

Figure 1: Triple pendulum with the datum at the origin O

Consider a planar pendulum consisting of three masses m_1, m_2, and m_3 suspended from massless rods of lengths L_1, L_2, and L_3 moving in the XY-plane (see Figure 1). An inertial frame of reference is fixed at the point of suspension, O, of the triple pendulum and the X-axis is taken as the datum for computing the potential energy of the system. Though simple, the system can exhibit complex dynamics.

The masses are constrained to move so that the total energy, $E(t)$, of the system is required to equal the sum of the energies (kinetic and potential) of only the two masses m_2 and m_3, i.e., $E(t) = E_2(t) + E_3(t)$ where we have denoted $E_i(t)$ as the total energy of mass m_i.

The three-step approach described in the last sub-section is now illustrated. We begin by writing the equation of the uncontrolled system (2.1) using the generalized coordinate 3-vector $q(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]^T$ whose components, in the absence of the above-mentioned energy control requirement, are independent of one another. Lagrange's equations then yield the relation

$$M(q;\ m_1, m_2, m_3)\ddot{q} = Q(q;\ m_1, m_2, m_3) \tag{2.10}$$

where the elements of the 3 by 3 symmetric matrix *M* are given by

$$\begin{aligned} &M_{11} = (m_1 + m_2 + m_3)L_1^2;\ M_{12} = (m_2 + m_3)L_1 L_2 \cos(\theta_{12}); \\ &M_{13} = m_3 L_1 L_3 \cos(\theta_{13}); M_{22} = (m_2 + m_3)L_2^2; \\ &M_{23} = m_3 L_2 L_3 \cos(\theta_{23});\ M_{33} = m_3 L_3^2, \end{aligned} \tag{2.11}$$

and the elements of the 3-vector *Q* are given by

$$\begin{aligned}Q_1 &= -(m_2+m_3)L_1L_2\dot{\theta}_2^2\sin(\theta_{12}) - m_3L_1L_3\dot{\theta}_3^2\sin(\theta_{13}) \\ &\quad -(m_1+m_2+m_3)gL_1\sin\theta_1 \\ Q_2 &= (m_2+m_3)L_1L_2\dot{\theta}_1^2\sin(\theta_{12}) - 2(m_2+m_3)L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_{12}) \\ &\quad -m_3L_2L_3\dot{\theta}_3^2\sin(\theta_{23}) - (m_2+m_3)gL_2\sin\theta_2 \\ Q_3 &= m_3L_1L_3(\dot{\theta}_1^2 - 2\dot{\theta}_1\dot{\theta}_3)\sin(\theta_{13}) + m_3L_2L_3(\dot{\theta}_2^2 - 2\dot{\theta}_2\dot{\theta}_3)\sin(\theta_{23}) \\ &\quad -m_3gL_3\sin\theta_3.\end{aligned} \tag{2.12}$$

In the above, we have denoted $\theta_{ij}(t) = \theta_i(t) - \theta_j(t)$, and we explicitly show in Eq. (2.10) the parameters m_1, m_2, and m_3 which we will later on consider to be known only imprecisely.

Using the X-axis as the datum, in the second step we describe the energy control requirement $E(t) = E_2(t) + E_3(t)$ which is equivalent to relation

$$E_1(t) = 0, \tag{2.13}$$

where the energy E_1 of mass m_1 is given by

$$E_1 = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 - m_1gL_1\cos\theta_1. \tag{2.14}$$

Since the system may not initially (at time *t* = 0) satisfy this control requirement we modify the control requirement (2.13) using the trajectory stabilization relation [7],

$$\dot{E}_1 + \alpha E_1 = 0, \tag{2.15}$$

where $\alpha(t) > 0$ is a positive function. By (2.14) and (2.15) we obtain the control requirement

$$A\ddot{q} := \begin{bmatrix} L_1^2\dot{\theta}_1 & 0 & 0 \end{bmatrix}\ddot{q} = -gL_1\sin\theta_1\dot{\theta}_1 - \alpha(\frac{1}{2}L_1^2\dot{\theta}_1^2 - gL_1\cos\theta_1) := b. \tag{2.16}$$

For the final step to obtain the equations of motion of the controlled nominal system we use the information from Eqs. (2.10)-(2.12) and (2.16) in Eq. (2.7). Pre-multiplying both sides of the equation by M^{-1}, we obtain the constrained acceleration of the (controlled) nominal system as (see Eq. (2.9)),

$$\ddot{q} = a + M^{-1}A^T(AM^{-1}A^T)^+(b - Aa). \tag{2.17}$$

2.3 Numerical results and simulations of the control problem

In what follows we shall assume that the real-life triple pendulum described above has masses whose values are imprecisely known, and that our best assessment of their values is: $m_1 = 1$kg, $m_2 = 2$kg, and $m_3 = 3$kg. Thus, these are the values of the three masses of our nominal system.

The lengths of the massless rods are $L_1 = 1$m, $L_2 = 1.5$m, and $L_3 = 2$m. At *t* = 0, the masses are located with the angles of $\theta_1(0) = 1$ rad, $\theta_2(0) = 0$ rad, and $\theta_3(0) = 0$ rad with respect to the vertical Y-axis (see Figure 1). The initial velocities of the three bobs are taken to be $\dot{\theta}_1(0) = 0.001$ rad/s, $\dot{\theta}_2(0) = 0$ rad/s, $\dot{\theta}_3(0) = 0$ rad/s. We note that these initial conditions do not satisfy the constraint, $E_1 = 0$. Thus the parameter α in (2.15) is chosen to be $0.02\|A\|_2^4$ where $\|A\|_2$ is the L^2 norm of the matrix *A* in (2.16). The acceleration due to gravity is downwards, and of magnitude $g = 9.81\, m/s^2$. Numerical integration throughout this paper is done in the *Matlab* environment, using a variable time step integrator with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}.

Figure 2 plots the trajectory of mass m_3 of the triple pendulum in the XY-plane for a period of 10 seconds. The start of the trajectory is marked by a circle and its end is marked by a square. From here on throughout this paper, the start and the end of all trajectories are indicated likewise. The energies of the three masses are shown in Figure 3. We see that the total energy (E) is the sum of the energies of mass m_2 (E_2) and mass m_3 (E_3), i.e. $E = E_2 + E_3$. Figure 3(a) also shows the extent of error in satisfying this control requirement $E_1 = 0$. The magnitude of this error is seen to be commensurate with the relative error tolerance used in the numerical integration. In Figure 4, we show the control force Q^c (2.8) on the nominal system in order to follow the desired control requirement $E_1 = 0$. Since only the first element of the matrix *A* in (2.16) is non-zero, the control forces on masses m_2 and m_3 are zero since the right-hand side of (2.8) is the product of A^T with a scalar. Figure 4(a) shows the control force required to be applied to mass m_1 to satisfy the constraint given in (2.15), and Figure 4(b) shows its magnitude.

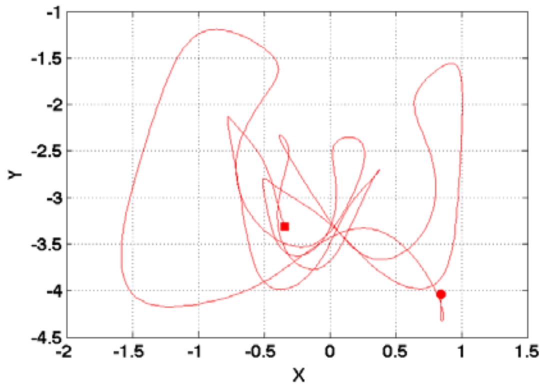

Figure 2: Trajectory of mass m_3 (meter)

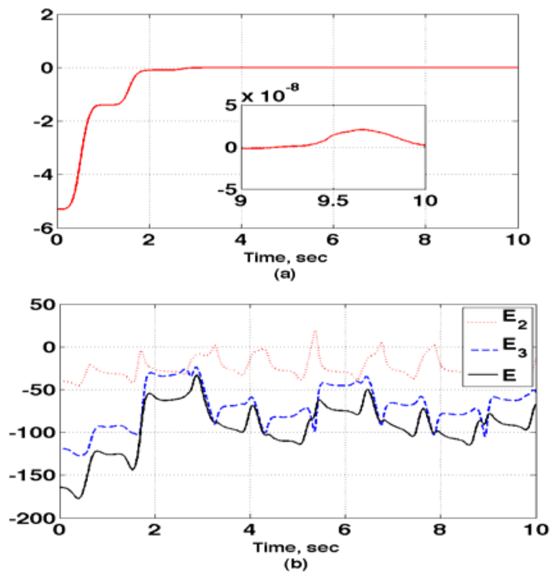

Figure 3: Energies in Newtons (a) E_1, (b) E = E_2+E_3

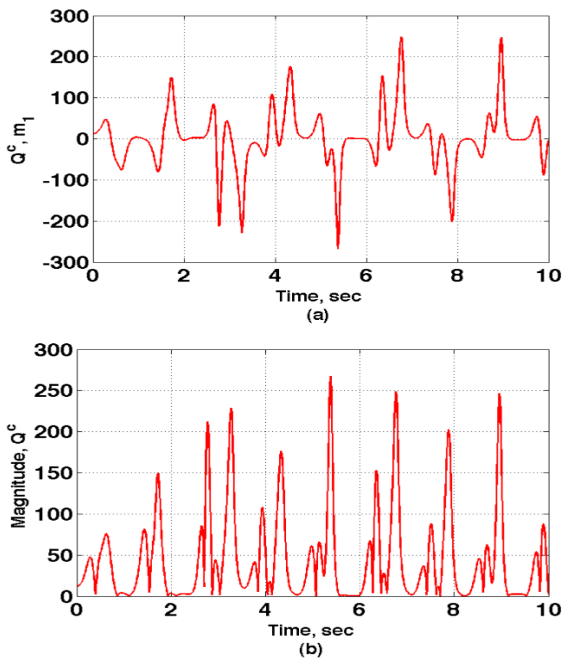

Figure 4: (*a*) Control force applied to mass m_1 of the nominal system to satisfy $E=E_2+E_3$ (*b*) Magnitude of the control force.

3. Description of the Control Approach

As mentioned before, there are always uncertainties in the description of any real-life dynamical systems. These uncertainties arise due to our lack of precise knowledge of the system, and/or of the 'given' forces acting on it. With the conceptualization of the nominal system given in the previous section, these uncertainties are now assumed to be encapsulated in the elements of the *n* by *n* matrix *M* and/or the *n*-vector *Q* (see Eq. (2.1)).

We assume that the mass matrix of the uncertain real-life system, which we do not know exactly, is $M_a := M + \delta M > 0$, where $M > 0$ is the *n* by *n* nominal mass matrix—our best estimate of the mass matrix of the actual system—, and δM is the *n* by *n* matrix that characterizes our uncertainty in the mass matrix of the actual system. The subscript '*a*' denotes the *actual*, real-life system whose knowledge is uncertain. Similarly, the 'given' force *n*-vector acting on the real-life system is taken to be $Q_a := Q + \delta Q$, where the *n*-vector Q denotes the 'given' force on the nominal system, and δQ denotes the *n*-vector of uncertainty in Q.

Our aim is to control this 'actual system' so that it mimics the motion of the nominal system and thereby satisfies the control requirements (2.4) imposed on the nominal system. With no exact knowledge of δM and δQ, the only control force that we have at hand to satisfy the control requirement (2.4) is the one we have obtained for the nominal system—our best estimate of the actual system, $Q^c(t)$. However, this control force is predicated on the perfect knowledge of the system and the use of an accurate model. Thus by applying this control force to the actual system, this causes the trajectories of the actual system and the nominal system to differ with a corresponding error in satisfaction of our desired control requirements (2.4).

To compensate for the uncertainty, the control force, $Q^c(t)$, needs to be modified since it was calculated on the basis of the nominal system and is now instead being applied to the actual unknown system. We do this by adding another control force Q^u from a compensating controller, resulting in a new state $(q_c, \dot{q}_c)$ (see Figure 5). We define the difference between $q_c(t)$ and $q(t)$ as the tracking error $e(t)$ (see Figure 5). In this paper, we develop this additive controller based on a generalization of the notion of a nonlinear damping, which is discussed in Section 5. A broad introduction to nonlinear damping control may be found in Ref. 11.

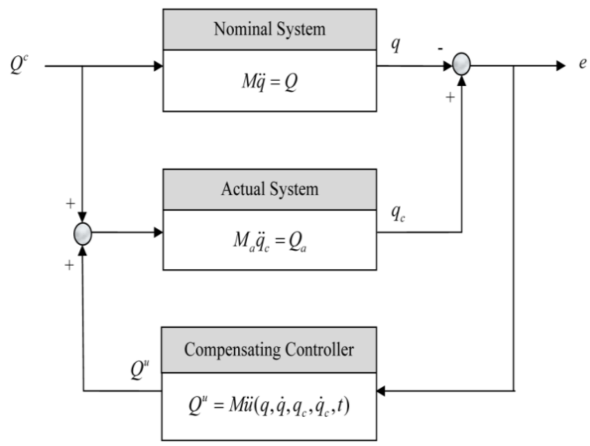

Figure 5: The block diagram of the controlled actual system.

The equation of motion of the controlled actual system thus becomes

$$M_a(q_c,t)\ddot{q}_c = Q_a(q_c,\dot{q}_c,t) + Q^c(t) + Q^u \quad (3.1)$$

where q_c is the generalized coordinate n-vector of the controlled actual system, $Q^c(t)$ is the control force which is obtained from the corresponding nominal system and which causes the nominal system to satisfy the constraint (2.5), and Q^u is the additional control force n-vector which we shall develop in closed form. We now refer to Eq. (3.1) as the description of the '*controlled actual system*,' or '*controlled system*,' for short. Pre-multiplying both sides of (3.1) by M_a^{-1}, the acceleration of this controlled system can then be expressed as

$$\ddot{q}_c = a_a + M_a^{-1} Q^c(t) + M_a^{-1} M \ddot{u}. \quad (3.2)$$

Here $a_a := M_a^{-1} Q_a$ and $Q^u := M\ddot{u}$, where $\ddot{u}$ is the additional generalized acceleration provided by the additional control forces Q^u to compensate for uncertainties in our knowledge of the actual system.

Before embarking on the determination of Q^u, we consider the uncertainties in the dynamics of the mechanical system next.

4. Uncertainties in the Dynamics of Mechanical Systems

Defining the tracking error as

$$e(t) = q_c(t) - q(t) \quad (4.1)$$

and differentiating (4.1) twice with respect to time, we get

$$\ddot{e} = \ddot{q}_c - \ddot{q}, \quad (4.2)$$

which upon use of (2.9) and (3.2) yields

$$\ddot{e} = \left[a_a(q_c,\dot{q}_c,t) - a(q,\dot{q},t)\right] + \left[M_a^{-1}(q_c,t) - M^{-1}(q,t)\right]Q^c(t) + M_a^{-1}M\ddot{u}$$
$$:= \delta\ddot{q} + M_a^{-1}M\ddot{u} = \delta\ddot{q} + [I - (I - M_a^{-1}M)]\ddot{u} := \delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u}. \quad (4.3)$$

In the above equation, we have defined

$$\bar{M} = I - M_a^{-1}(q_c,t)\, M(q,t) = I - \left(M(q_c,t) + \delta M(q_c,t)\right)^{-1} M(q,t)$$
$$= I - \left(M^{-1}(q,t)\, M(q_c,t) + M^{-1}(q,t)\ \delta M(q_c,t)\right)^{-1}, \quad (4.4)$$

and denoted the acceleration $\delta\ddot{q}$ as

$$\delta\ddot{q}(q,\dot{q},q_c,\dot{q}_c,t) = \left[a_a(q_c,\dot{q}_c,t) - a(q,\dot{q},t)\right] + \left[M_a^{-1}(q_c,t) - M^{-1}(q,t)\right]Q^c(t), \quad (4.5)$$

where $a_a := M_a^{-1}Q_a$, with $M_a := M(q_c,t) + \delta M(q_c,t)$, and $Q_a := Q(q_c,\dot{q}_c,t) + \delta Q(q_c,\dot{q}_c,t)$.

5. Generalized Nonlinear Damping Controller

Our aim in this section is to develop a set of compensating controllers that can guarantee the tracking of the nominal system's trajectory despite our uncertain knowledge of the actual system. To do this we use a generalization of the concept of a nonlinear damping [9]. The formulation permits the use of a large class of control laws that can be adapted to the practical limitations of the specific compensating controller being used, and the extent to which we want to compensate for the uncertainties.

Noting (4.3), the tracking error signal in acceleration can be expressed as

$$\ddot{e} = \delta\ddot{q} + M_a^{-1}M\ddot{u} := \delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u}. \quad (5.1)$$

The aim is to develop a controller $\ddot{u}$ such that the motion of the controlled actual system closely tracks the motion of the nominal system. We assume for the moment that the compensating control acceleration $\ddot{u}$ is capable of this and causes the trajectory of the controlled actual system $(q_c,\dot{q}_c)$ to sufficiently approximate that of the nominal system so that $(q_c,\dot{q}_c) \approx (q,\dot{q})$. Under this assumption, we take the lowest order terms in Eq. (4.4) and approximate $\bar{M}$ as

$$\bar{M} \approx I - \left(I + M^{-1}(q,t)\ \delta M(q,t)\right)^{-1}. \quad (5.2)$$

We note that $\bar{M}$ is unknown, since δM is unknown, and it is embedded in our controller $\ddot{u}$ (see Eq. (5.1)). We shall show that the uncertain term $\bar{M}$ will be taken care of by the proof of Lyapunov stability.

We start by defining the system tracking error between the nominal and the controlled actual systems in the state space form as (see Eq. (5.1))

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\end{bmatrix} = \begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} + \begin{bmatrix}0\\ I\end{bmatrix}\left(\delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u}\right), \quad (5.3)$$

where $e_1 := e,\ e_2 := \dot{e}$, and $\bar{M} \approx I - \left(I + M^{-1}\delta M\right)^{-1}$.

Consider the Lyapunov candidate function

$$V = \frac{1}{2}e_1^T e_1 + \frac{1}{2}e_2^T e_2. \quad (5.4)$$

Differentiating Eq. (5.4) once with respect to time and using Eq. (5.3), we get

$$\begin{aligned}\dot{V} &= e_1^T \dot{e}_1 + e_2^T \dot{e}_2 \\ &= e_1^T e_2 + e_2^T (\delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u}).\end{aligned} \tag{5.5}$$

In order to guarantee the Lyapunov stability of the system (5.3), we shall now show that the derivative $\dot{V}$ is negative. We then start by considering the controller, which is of the form

$$\ddot{u} = -\left(e_1(t) + f(e_2)\right) - ke_2(t), \tag{5.6}$$

where $k > 0$ is arbitrary small positive constants. The *i*-th component, $f_i(e_2)$, of the *n*-vector $f(e_2)$ is defined as

$$f_i(e_2) = g\left(e_{2,i}/\varepsilon\right),\ i = 1, \ldots, n \tag{5.7}$$

where $e_{2,i}$ is the *i*-th component of the *n*-vector e_2, ε is defined as any small positive number and the function $g\left(e_{2,i}/\varepsilon\right)$ is *any* arbitrary strictly increasing odd continuous function of $e_{2,i}$ on the interval $(-\infty, +\infty)$ and it goes to ∞ as $e_{2,i}$ goes to ∞.

Result 1: The generalized damping controller

$$\ddot{u} = -\left(e_1(t) + f(e_2)\right) - ke_2(t) \tag{5.8}$$

guarantees that the Lyapunov derivative $\dot{V}$ is negative. Thus the solution of the closed-loop system (5.3) is uniformly bounded.

Proof: Using the controller (5.6) in (5.5), we have

$$\dot{V} = e_2^T \delta\ddot{q} - e_2^T f(e_2) - ke_2^T e_2 + e_2^T \bar{M} e_1 + e_2^T \bar{M} f(e_2) + ke_2^T \bar{M} e_2. \tag{5.9}$$

Since

$$e_2^T f(e_2) \geq \|e_2\|_\infty \|f(e_2)\|_\infty, \tag{5.10}$$

using Eq. (5.10) in Eq. (5.9), we obtain

$$\begin{aligned}\dot{V} \leq{}& \|e_2^T\|_\infty \|\delta\ddot{q}\|_\infty - \|e_2\|_\infty \|f(e_2)\|_\infty - k\|e_2\|_2^2 \\ &+ \|e_2^T\|_\infty \|\bar{M}\|_\infty \|e_1\|_\infty + \|e_2^T\|_\infty \|\bar{M}\|_\infty \|f(e_2)\|_\infty + k\|e_2^T\|_\infty \|\bar{M}\|_\infty \|e_2\|_\infty.\end{aligned} \tag{5.11}$$

We assume

$$\|M^{-1}\delta M\|_\infty << 1, \tag{5.12}$$

thus,

$$\|\bar{M}\|_\infty \approx \left\|I - \left(I + M^{-1}\ \delta M\right)^{-1}\right\|_\infty \approx \|M^{-1}\ \delta M\|_\infty. \tag{5.13}$$

And since $\|\delta M\|_\infty \|e_1\|_\infty \approx \|\delta M\|_\infty \|e_2\|_\infty \approx \underline{0}$, and also k is any small positive number so that the term

$$\|e_2^T\|_\infty \|\bar{M}\|_\infty \|e_1\|_\infty \approx k\|e_2^T\|_\infty \|\bar{M}\|_\infty \|e_2\|_\infty \approx 0, \tag{5.14}$$

thus Eq. (5.11) yields

$$\begin{aligned}\dot{V} &\leq \|e_2^T\|_\infty \|\delta\ddot{q}\|_\infty - \|e_2\|_\infty \|f(e_2)\|_\infty - k\|e_2\|_2^2 + \|e_2^T\|_\infty \|\bar{M}\|_\infty \|f(e_2)\|_\infty \\ &\leq \|e_2^T\|_\infty \Gamma - \|e_2\|_\infty \|f(e_2)\|_\infty - k\|e_2\|_\infty^2 + \|e_2^T\|_\infty \|\bar{M}\|_\infty \|f(e_2)\|_\infty \\ &\leq -\|e_2\|_\infty \left(1 - n\|\bar{M}\|_\infty\right) \|f(e_2)\|_\infty - k\|e_2\|_\infty^2 + n\|e_2\|_\infty \Gamma.\end{aligned} \tag{5.15}$$

The second inequality above follows from $\|e_2\|_2 \geq \|e_2\|_\infty$ and we denote $\|\delta\ddot{q}\| \leq \Gamma$, where Γ is an upper bound on $\|\delta\ddot{q}(t)\|$ (see Eq. (4.5)) and the third inequality follows because $n\|e_2\|_\infty \geq \|e_2^T\|_\infty$.

We note that the term $-k\left\|e_2\right\|_\infty^2 + n\left\|e_2\right\|_\infty \Gamma$ above attains a maximum value $\frac{n^2\Gamma^2}{4k}$ at $\left\|e_2\right\|_\infty = \frac{n\Gamma}{2k}$. Therefore

$$\dot{V} \le -\left\|e_2\right\|_\infty \left(1 - n\left\|\bar{M}\right\|_\infty\right)\left\| f(e_2)\right\|_\infty + \frac{n^2\Gamma^2}{4k}. \quad (5.16)$$

From Eqs. (5.12) and (5.13), if we further assume that $n\left\|\bar{M}\right\|_\infty << 1$ and since $f(e_2)$ is a strictly increasing function of e_2, which goes to ∞ as e_2 goes to ∞, it is always true that $\dot{V}$ is negative outside some ball. Thus the solution of the closed-loop system (Eq. (5.3)) is uniformly bounded [9]. □

<u>Main Result</u>: The closed-from generalized damping controller for the uncertain system,

$$\begin{aligned} M_a \ddot{q}_c &= Q_a + Q^c(t) + M\ddot{u} \\ &= Q_a + Q^c(t) - M\left[\left(e_1(t) + f(e_2)\right) + ke_2(t)\right], \end{aligned} \quad (5.17)$$

where:

(i) the control force $Q^c(t)$ is given by (2.8)

$$Q^c(t) = A^T(AM^{-1}A^T)^+(b - Aa) \quad (5.18)$$

and is obtained on the basis of the nominal system;

(ii) $k > 0$ is arbitrary small positive constant; and

(iii) $f(e_2)$ is *any* arbitrary strictly increasing odd continuous function of e_2 on the interval $(-\infty, +\infty)$ and goes to ∞ as e_2 goes to ∞,

will cause the actual system to track the trajectory of the nominal system.

6. Numerical Results and Simulations

In this section we continue to illustrate the control methodology in the presence of uncertainties by considering the same example of the triple pendulum. The approach is straightforward to apply to other systems. While our nominal system has $m_1 = 1, m_2 = 2,$ and $m_3 = 3$, there is an uncertainty of up to ± 10% in each of these values when describing the actual system.

With imperfect knowledge of the parameters in the system, in order to control the actual system's motion so that it tracks the motion of the controlled nominal system, and thereby satisfies the control requirements imposed on the nominal system, we would have to use Eq. (5.17) that contains the additional controller to compensate for our uncertainty in the knowledge of the actual system.

We next select the structure and parameters for the controller $\ddot{u}$ given by Eq. (5.8). We choose

$$f_i(e_2) = \alpha_c (e_{2,i}/\varepsilon)^3, \quad (6.1)$$

where $\alpha_c, \varepsilon > 0$ and ε is a suitable small number. We then obtain in closed-form the additional controller needed to compensate for uncertainties in the actual system as

$$\ddot{u}_i(t) = -\left(e_i(t) + k\dot{e}_i(t)\right) - \alpha_c(\dot{e}_i/\varepsilon)^3. \qquad (6.2)$$

Pre-multiplying both sides of Eq. (5.17) by M_a^{-1} and using the additional controller Eq. (6.2), we obtain the closed-form equation of motion of the controlled actual system as

$$\ddot{q}_c = a_a + M_a^{-1}Q^c(t) - M_a^{-1}M\left[\left(e_i(t) + k\dot{e}_i(t)\right) + \alpha_c(\dot{e}_i/\varepsilon)^3\right] \qquad (6.3)$$

that will cause the actual system to track the trajectory of the nominal system, thereby compensating for the uncertainty in our knowledge of the actual system.

In order to illustrate the efficacy of our controller in compensating for our lack of exact knowledge of the actual system in the presence of the ± 10 percent uncertainties in each of the masses m_1, m_2, and m_3, we pick the set $\delta m_1 = 0.1,$ $\delta m_2 = -0.2,$ and $\delta m_3 = 0.3$, which is assumed to represent our actual system. We note that the elements of the 3 by 3 symmetric matrix M_a and of the 3-vector Q_a are given in a manner similar to Eqs. (2.11) and (2.12) respectively. In this case, we have replaced m_i in (2.11) and (2.12) with $m_i = m_i + \delta m_i,\ i = 1,2,3$. We note that $A_a = A$ and $b_a = b$, since our constraint (2.16) does not involve any of the masses m_i.To check the performance of our controller, we perform a simulation using Eq. (6.3) with the parameters $k = 10,\ \alpha_c = 2,$ and $\varepsilon = 10^{-4}$, to specify our controller. All the other parameter values are the same as those prescribed in Section 2.3.

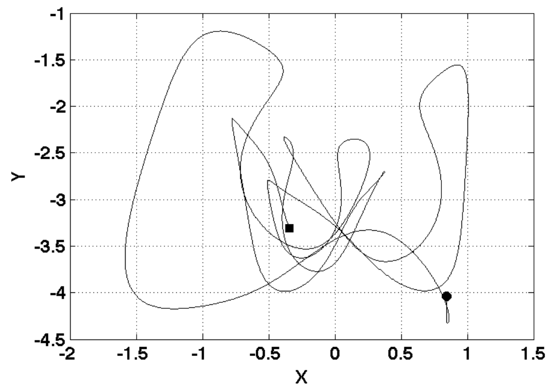

Figure 6: Trajectory response (meter) of the mass m_3 of the controlled actual system

The constrained trajectory of mass m_3 in the XY-plane of the controlled actual system is illustrated in Figure 6. We see that the controlled system (given by (6.3) and shown in Figure 6) tracks the nominal system (given by (2.17) and shown in Figure 2). We note that both systems satisfy the energy control requirement (2.15). This illustrates the performance of the closed-form Eq. (6.3) showing that the controlled actual system tracks the trajectories pre-specified by the nominal system in the presence of the $\pm 10\%$ uncertainties in masses of the triple pendulum and the control requirement imposed on it given by Eq. (2.15). Figure 7 and Figure 8 correspondingly show the displacement errors $(q - q_c)$ and the velocity errors $(\dot{q} - \dot{q}_c)$ between the nominal system (2.17) and the controlled actual system (6.3). Both tracking errors are small, which are seen to be of

$O(10^{-5})$ and of $O(10^{-4})$ respectively. We note that use of the smooth cubic function $f_i(e_2)$ given in (6.1) eliminates chattering.

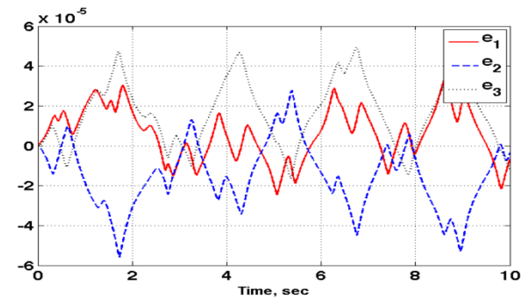

Figure 7: Displacement errors
$(e_i(t) := \theta_i(t) - \theta_{c_i}(t),\ i = 1,2,3)$ in radians

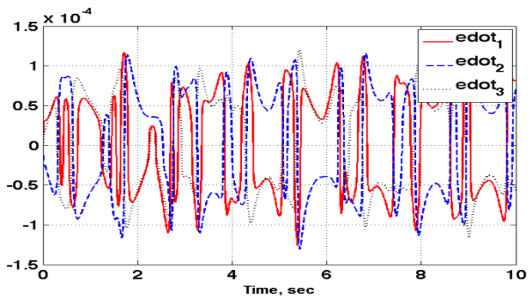

Figure 8: Velocity errors
$(\dot{e}_i(t) := \dot{\theta}_i(t) - \dot{\theta}_{c_i}(t),\ i = 1,2,3)$ in radian/sec

Pre-multiplying (6.3) by M_a, we obtain (see (5.17))

$$M_a \ddot{q}_c = Q_a + Q^c - M\left[\left(e_i(t) + k\dot{e}_i(t)\right) + \alpha_c(\dot{e}_i/\varepsilon)^3\right]$$
$$:= Q_a + Q^c + M\ddot{u} := Q_a + Q^c + Q^u. \qquad (6.4)$$

The total control force applied to the actual system is given by $Q^T = Q^c + Q^u$. Here Q^c is the control force obtained from the nominal system, and Q^u is the force applied by the additional compensating controller to compensate for our inexact knowledge of the actual system. The control forces Q^T and Q^u on the masses m_1, m_2, and m_3 of the actual pendulum are shown in Figure 9. The magnitude of the additional control forces, Q^u, applied by the compensating controller is seen to be small relative to the magnitude of the total control forces, Q^T.

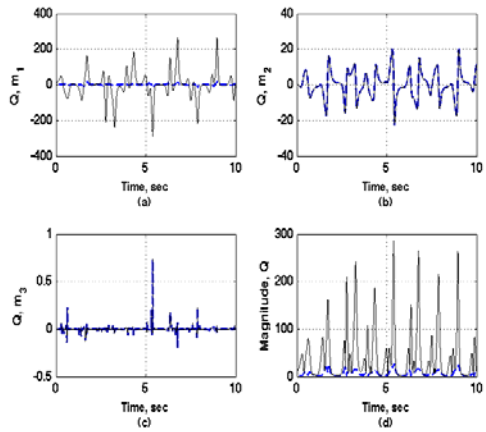

Figure 9: Control forces (Newtons) on the controlled actual system. The solid line show the total control force, Q^T, the dashed line shows the additional force Q^u, needed to compensate for uncertainties in the actual system

7. Conclusion

In this paper, a set of closed-form controllers for uncertain nonlinear multi-body systems is developed. The main contributions of this paper are the following:

(i) We obtain the exact closed-form solution to the energy control problem of a multi-body system. The control force that must be applied to the system because of the presence of the energy control requirement imposed on the system is easily obtained. Also, when starting with initial states that do not satisfy this energy requirement, the error in satisfying it converges to zero exponentially.

(ii) The general closed-form equation of motion for uncertain nonlinear multi-body systems—the so-called *controlled actual system*—has been developed. The novelty in the approach developed here is that we first use the *fundamental equation* to obtain exact control of the nominal, nonlinear, non-autonomous, mechanical system. This control, Q^c, ensures that the trajectory constraints are exactly satisfied by the nominal system and that it optimizes the control cost given by $Q^{c^T} M^{-1} Q^c$ at *each* instant of time. More general control costs can also be considered as in Ref. [7]. Control of the *actual system*, in which both the mass matrix and the 'given' forces may be only imprecisely know, is then carried out using the concept of generalized nonlinear damping.

(iii) We have generalized the concept of a nonlinear damping by including functions $f_i(e_2)$ that are not necessarily signum or saturation functions [9]. The control functions $f_i(e_2)$ and the parameters that define the compensating controller can therefore be chosen depending on practical considerations of the control environment and on the extent to which the compensation of the uncertainties is desired. Thus when dealing with large, complex multi-body systems greater flexibility is afforded. For example, the use of a cubic function may obviate the need for a high-gain controller and would also allow continuous control, thereby preventing chattering.

(iv) The formulation of the proposed control methodology encompasses both general sources of uncertainties—uncertainties in the description of the physical system and uncertainties in knowledge of the 'given' forces applied to the system. The set of closed-form controllers developed herein is therefore general enough to be applicable to complex dynamical systems in which uncertainties of both these types may arise.

8. References

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