Development of 3D Finite Element Software for Free Vibration Analysis without Damping

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Abstract

In order to avoid mechanical resonance, a vibrating structure needs to be designed such that its working frequency interval is sufficiently far from its natural frequency. Natural frequency of a mechanical system is obtained from free vibration analysis commonly done by finite element method (FEM). In this work, we demonstrate the development of 3D FEM computer software for free vibration analysis without damping which is an eigenvalue problem. The governing FEM equations had been derived in the standard form of eigenvalue problem before matrix tridiagonalization was performed. Three tridiagonalization methods including the Lanczos method, the Householder method and the block Lanczos method were chosen for comparison purposes. The QL algorithm with implicit shifts was implemented to solve the derived FEM equations. The computational performances of the developed FEM programs have been evaluated by several free vibration problems. The simulation results show that the Householder and the block Lanczos methods are more suitable for free vibration analysis than the Lanczos method.

Keywords*:* Free vibration; Finite element; Tridiagonalization; Householder; Block Lanczos

1. Introduction

One of the most common problems for a mechanical system is mechanical resonance. When the resonance occurs, the structure oscillates with higher amplitude than it does at other frequencies. A vibrating system is prone to mechanical resonance when the frequency of its oscillations matches the system's natural frequency. Mechanical resonance can cause unwanted violent motions and even structural failure in structures including buildings, bridges, machines and vehicles. Therefore, design engineers must ensure that the working frequency interval of the mechanical system is sufficiently far from its natural frequency.

Natural frequency can be determined from free vibration analysis [1,2]. For 3D complex geometry, this is commonly done by finite element method (FEM). In this paper, the development of 3D FEM computer software for free vibration analysis without damping, which is an eigenvalue problem [3], is demonstrated. A

comparison between three tridiagonalization methods [4], which are Lanczos [5], Householder [4,6] and block Lanczos [7,8], was made to find suitable tridiagonalization methods for free vibration analysis. The QL algorithm with implicit shifts [6,9] was employed to solve the derived FEM equations. The performances of the developed FEM computer programs have been evaluated by several problems having exact solutions and/or experimental results. Furthermore, computational results by ANSYS™ [10], a widely–used commercial FEM software package, have also been shown for comparison purposes.

2. Finite Element Derivation

2.1 Finite Element Equations

The governing equation for 3D free vibration analysis without damping is:

$$m\left(\frac{d^2u}{dt^2}+\frac{d^2v}{dt^2}+\frac{d^2w}{dt^2}\right)+k(u+v+w)=0 \quad (1)$$

where m is mass, k is stiffness coefficient, t is time, and u, v and w are displacement in x, y and z directions, respectively. By FEM derivation, the partial differential equation (Eq. (1)) is transformed into algebraic equation [11]:

$$[M]\{\ddot{\delta}\}+[K]\{\delta\}=0 \quad (2)$$

where $[M]$ is mass matrix,

$[K]$ is stiffness matrix and

$\{\delta\}$ is displacement matrix.

The oscillation is assumed to be simple harmonic motion [12] which can be described by:

$$\{\delta\}=\{\phi\}\sin\omega t \quad (3)$$

where $\{\phi\}$ is amplitude of motion and

ω is angular frequency.

Angular frequency is related to frequency f by:

$$\omega=2\pi f \quad (4)$$

The second derivative of Eq. (3) is:

$$\{\ddot{\delta}\}=-\omega^2\{\phi\}\sin\omega t \quad (5)$$

By substituting Eqs. (3) and (5) into Eq. (2), we obtain:

$$\left[[K]-\omega^2[M]\right]\{\phi\}=0 \quad (6)$$

where

$$[M]=\int_V \rho[N]^T[N]dV \quad (7)$$

here, ρ is density, and

$$[K]=\int_V [B]^T[C][B]dV=[B]^T[C][B]V \quad (8)$$

where, $[C]$ is stress-strain relation matrix,

$$[B]=[L][N] \quad (9)$$

$[L]$ is strain-displacement relation matrix

$[N]$ is interpolation function matrix.

2.2 Tetrahedral Element

The four-node tetrahedral element type [13] was used this study. The distribution of variables can be written as:

$$\phi=N_1\phi_1+N_2\phi_2+N_3\phi_3+N_4\phi_4=\lfloor N\rfloor\{\phi\} \quad (10)$$

For 3D free vibration analysis, the displacement matrix can be written as:

$$\{\delta\}=[N]\{\bar{\delta}\} \quad (11)$$

where

$$\{\delta\}^T=\lfloor u \quad v \quad w\rfloor \quad (12)$$

$$\{\bar{\delta}\}^T=\lfloor u_1 \quad v_1 \quad w_1 \quad \cdots \quad u_4 \quad v_4 \quad w_4\rfloor \quad (13)$$

$$[N]=\begin{bmatrix} N_1 & 0 & 0 & \cdots & N_4 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_4 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_4 \end{bmatrix} \quad (14)$$

3. Numerical Methods

The procedure for solving derived FEM free vibration equations contains three steps:

1. Transforming the set of equations into standard-form matrix [14]
2. Tridiagonalizing the equation matrix
3. Determining eigenvalues and eigenvectors

Therefore, the procedure of free vibration analysis by FEM can be schematically shown in Fig. 1.

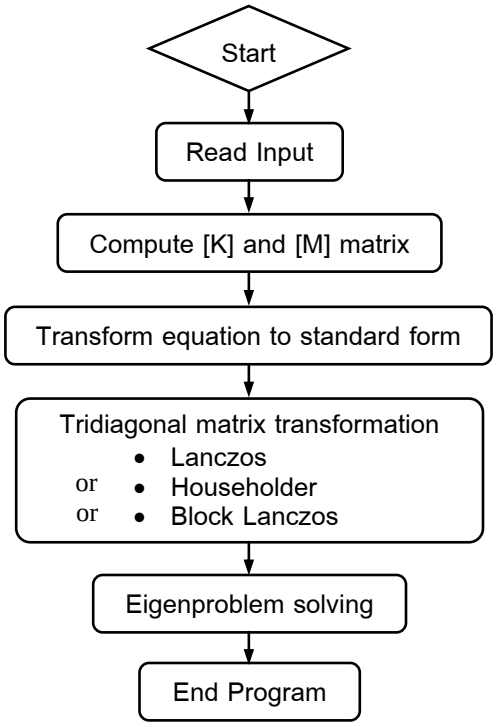

Fig. 1 Procedure of FEM free vibration analysis

3.1 FEM Equations in Standard Form

Eq. (6) can be transformed into standard form of the eigenvalue problem as:

$$\left[[A]-\lambda[I]\right]\{\psi\}=0 \tag{15}$$

where $\lambda=\omega^2$ and $[I]$ is identity matrix.

Mass matrix in Eq. (6) can be written as:

$$[M]=[L][L]^T \tag{16}$$

where $[L]$ is a lower triangular matrix and can be determined using the Cholesky's symmetric decomposition.

Substituting Eq. (16) into Eq. (6) leads to:

$$\left[[K]-\lambda[L][L]^T\right]\{\phi\}=0 \tag{17}$$

Premultiplying Eq. (17) with $[L]^{-1}$ and substituting:

$$\{\phi\}=[L]^{-T}\{\psi\} \tag{18}$$

into Eq. (17) leads to:

$$\left[[L]^{-1}[K][L]^{-T}-\lambda[I]\right]\{\psi\}=0 \tag{19}$$

Therefore, $[A]$ of the standard form equation (Eq. (15)) for Eq. (6) is:

$$[A]=[L]^{-1}[K][L]^{-T} \tag{20}$$

3.2 Tridiagonalization

Tridiagonal matrix is a matrix which has only nonzero values on the main diagonal, and on the upper and lower subdiagonals. In FEM analysis for eigenvalue problems, it is common to transform matrices into the tridiagonal form. This leads to the reduction in computational effort required while the eigenvalues of the original matrices are still preserved. There are several methods for the matrix tridiagonalization operation. In this paper, we consider three methods including the Lanczos method, the Householder method and the block Lanczos method.

3.2.1 The Lanczos Method

The Lanczos algorithm was introduced by C. Lanczos in 1950 [15]. Lanczos intended his algorithm to be used for computing a few of the extreme eigenvalues and corresponding eigenvectors of a symmetric matrix. However, the algorithm was taken up as a method for reducing a symmetric matrix to tridiagonal form [16]. A real symmetric matrix can define corresponding Lanczos matrix using the recursion as shown in Eqs. (21) – (23):

$$\beta_{i+1}\{v_{i+1}\}=[A]\{v_i\}-\alpha_i\{v_i\}-\beta_i\{v_{i-1}\} \tag{21}$$

$$\alpha_i=\{v\}_i^T[A]\{v\}_i \tag{22}$$

$$\beta_{i+1}=\{v\}_{i+1}^T[A]\{v\}_i \tag{23}$$

where $\{v\}_i$ is a Lanczos vector, α and β are scalar values.

The sequence started by letting $\{v\}_1$ be a unit starting vector (generally generated randomly).

Define β_1 = 0 and $\{v\}_0$ = 0. For $i = 1,2,3,...,j$, the corresponding Lanczos matrix is define as the tridiagonal matrix with diagonal entries α_i and subdiagonal entries β_i as shown below:

$$[T]_j = \begin{bmatrix} \alpha_1 & \beta_2 & & & 0 \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \beta_3 & \ddots & \ddots & \\ & & \ddots & \alpha_j & \beta_j \\ 0 & & & \beta_j & \alpha_j \end{bmatrix} \quad (24)$$

3.2.2 The Householder Method

The householder method was introduced in 1958 by A.S. Householder [17]. By applying a sequence of Householder reflection matrices, any symmetric matrix can be converted into tridiagonal form. Each sequence produces a complete row and column of zeros apart from the elements within the tridiagonal and subdiagonal. Householder reflection matrix can be written as:

$$[H] = [I] - 2\{u\}\{u\}^T \quad (25)$$

where $\{u\}$ is a unit vector:

$$\{u\} = \frac{\{x\} - \{y\}}{\|\{x\} - \{y\}\|} \quad (26)$$

Vector $\{x\}$ contains all the off-diagonal entries of each column of matrix $[A]$. Vector $\{y\}$ contains subdiagonal values of that column. For example,

$$\{x_1\}^T = \lfloor 0 \quad a_{21} \quad a_{31} \quad \cdots \quad a_{n1} \rfloor \quad (27)$$

$$\{y_1\}^T = \lfloor 0 \quad \pm r_1 \quad 0 \quad \cdots \quad 0 \rfloor \quad (28)$$

where $$r_1 = \|\{x_1\}\| = \|\{y_1\}\| \quad (29)$$

The sequence is computed by:

$$[A]_2 = [H]_1[A][H]_1$$

$$= \begin{bmatrix} a_{11} & r_1 & 0 & \cdots & 0 \\ r_1 & \tilde{a}_{22} & \tilde{a}_{22} & \cdots & \tilde{a}_{22} \\ 0 & \tilde{a}_{22} & \tilde{a}_{22} & \cdots & \tilde{a}_{22} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{a}_{22} & \tilde{a}_{22} & \cdots & \tilde{a}_{22} \end{bmatrix} \quad (30)$$

Thus, by a single Householder transformation, matrix $[A]$ is converted into a similar matrix $[A]_2$ whose first row and column are in tridiagonal form. By repeating the process on the lower right $(n-1)\times(n-1)$ submatrix of $[A]_2$, the next matrix $[A]_3$ whose first two rows and columns are in tridiagonal form are constructed. The process is repeated until the final result is a tridiagonal matrix.

3.2.3 The Block Lanczos Method

The idea of the block Lanczos algorithm is to change $[A]$ into a block tridiagonal matrix $[J]$ containing small square matrices on the diagonal and on the upper and lower subdiagonals, as shown in Eq. (31):

$$[J] = \begin{bmatrix} [D_1] & [B_1]^T & & 0 & 0 \\ [B_1] & [D_2] & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & [D_{p-1}] & [B_{p-1}]^T \\ 0 & 0 & & [B_{p-1}] & [D_p] \end{bmatrix} \quad (31)$$

From Eq. (31), to compute $[D_i]$ where $i = 1,2,...,p$, and $[B_j]$ where $j = 1,2,...,p-1$, we can use the block Lanczos algorithm [8]

3.3 Solvers for FEM Equations

After tridiagonalization, the QL with implicit shift algorithm was used to solve the FEM equations. The QL algorithm decomposes the matrix $[A]$ into a product of an orthogonal matrix $[Q]$ and a lower triangular matrix $[L]$. Therefore, the matrix $[A]$ can be expressed as:

$$[A]_n = [Q]_n[L]_n \quad (32)$$

Then calculate the new matrix $[A]$ by switching the order of $[Q]_n$ and $[L]_n$ in Eq.(32):

$$[A]_{n+1} = [L]_n[Q]_n \quad (33)$$

The sequence when $n \to \infty$, $[A]_{n+1}$ converges to a diagonal matrix which the eigenvalues appear on the diagonal.

Convergence can be accelerated by using the shifting technique. The technique can also be improved to implicit shift to avoid the loss of accuracy for the small eigenvalues.

4. Evaluation of the FEM Programs

Three problems were chosen for the evaluation of the three FEM computer programs using different tridiagonalization methods, which are Lanczos, Householder and block Lanczos. Mode frequencies (eigenvalues) obtained from the FEM programs have been compared with those obtained from ANSYS™ and exact solutions (or experimental results).

4.1 Solid Cube

A solid cube is fixed at its base as shown in Fig. 2 [18]. Material properties of the solid cube are Young's Modulus of 68.95×10^9 N/m^2, Poisson's ratio of 0.3 and density of 2,560 kg/m^3.

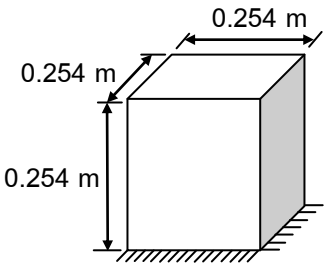

Fig. 2 Geometry of a solid cube

The computational results from the three FEM computer programs (Lanczos, Householder and block Lanczos) were compared with those from ANSYS and exact solutions. Both ANSYS and the developed programs provide the vibration modes corresponding with exact solutions (dash line) as shown in Figs. 3-6. These figures show that the mode frequencies obtained from the developed FEM programs and ANSYS are closer to exact solutions when node number increases.

For the solutions of mode frequency in mode 1, Fig. 3 shows that the difference between the computational value and the exact solution is 6.14% and 5.58% for ANSYS and the developed programs using 97 nodes, respectively. As node number increases, the computational solutions are closer to the exact solution. When using 1,000 nodes, ANSYS and the developed programs give the results with only 0.08% and 0.01% difference to the exact solution.

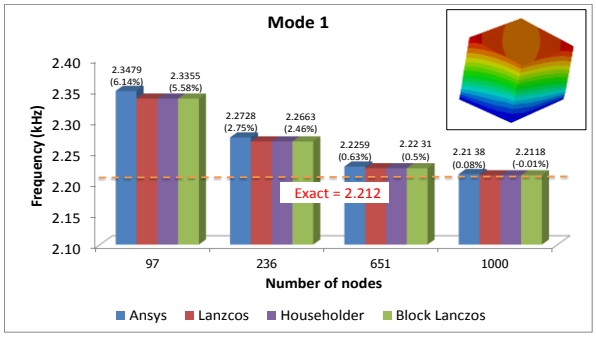

Fig. 3 Comparison of mode frequency values for mode 1 (swaying mode). The values above the columns are mode frequencies computed and differences with the exact solution are shown in parentheses.

Figs. 4-6 show that the discrepancy between the computed results and the exact solution in modes 3 to 5 is less than 2% when using 1,000 nodes. At 1,000 nodes, ANSYS and the developed FEM programs give very similar results (less than 0.5% difference). It is noted that solutions for mode 2 are not shown because there is no report of exact solution.

The developed FEM programs using the Householder and Lanczos methods give eigenvalue results with some repetition in values.

However, this is not a serious problem as users can ignore the repeated values. In contrast, Table 1 shows that the developed FEM program using the Lanczos method provides some computed values which are significantly different to the repeated values. Note that these results from the Lanczos method are not used in the result comparisons in Figs. 3-6.

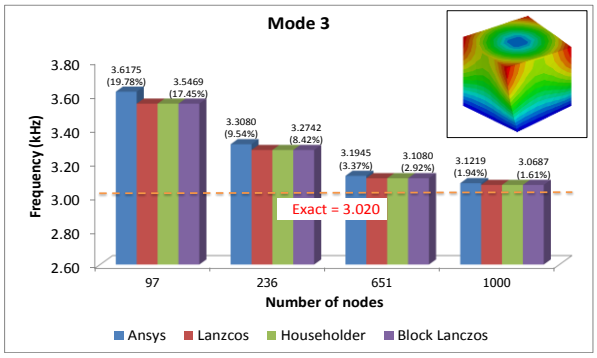

Fig. 4 Comparison of mode frequency values for mode 3 (torsion mode)

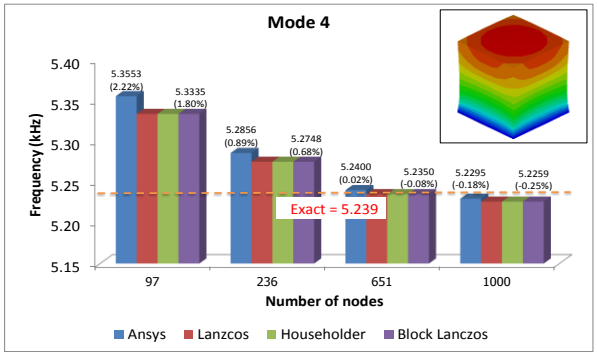

Fig. 5 Comparison of mode frequency values for mode 4 (longitudinal mode)

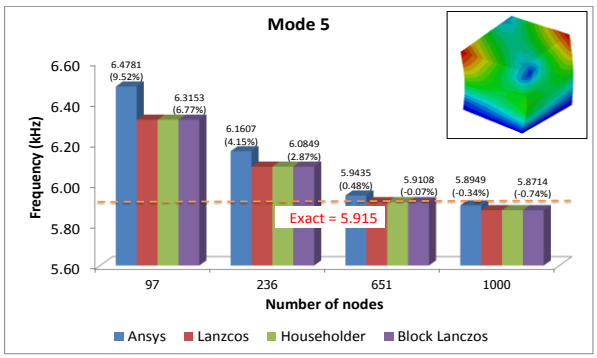

Fig. 6 Comparison of mode frequency values for mode 5 (swaying mode)

Table. 1 All frequency values computed by the FEM program using the Lanczos method. The erroneous values are highlighted by yellow color.

Modes	Number of nodes					
	67	97	236	423	651	1000
1	2.437833	2.335490	2.266373	2.241599	2.223123	2.211758
2	2.476365	2.342088	2.266373	2.241599	2.223123	2.211758
3	3.829661	2.363350	2.276090	2.241599	2.223123	2.211758
4	5.394683	3.546944	2.276090	2.249288	2.223123	2.211758
5	6.564503	3.547507	3.274194	2.249288	2.228007	2.211758
6		5.333476	3.274194	2.249288	2.228007	2.215306
7		6.315265	3.286584	3.174731	2.228007	2.215306
8			5.274799	3.174731	2.228007	2.215306
9			5.274799	3.174731	3.108044	2.215306
10			6.084888	5.250132	3.108044	2.215306
11			6.084888	5.250132	3.108044	3.068655
12				5.250132	3.108044	3.068655
13				5.969032	3.111897	3.068655
14				5.969032	5.235033	3.068655
15					5.235033	3.068655
16					5.235033	3.068655
17					5.235033	5.225902
18					5.910831	5.225902
19					5.910831	5.225902
20					5.910831	5.225902
21						5.225902
22						5.871402
23						5.871402
24						5.871402
25						5.871402

4.2 Cantilever Beam

A cantilever beam is fixed at one side as shown in Fig. 7 [18]. Material properties of the cantilever beam are Young's Modulus of 2.068×10^{11} N/m^2, Poisson's ratio of 0.3 and density of 8,058 kg/m^3.

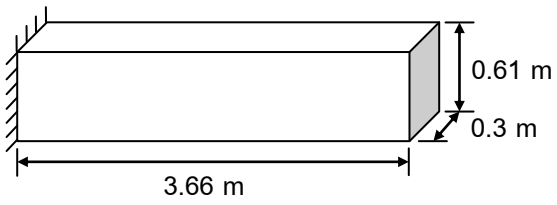

Fig. 7 Geometry of a cantilever beam

Figs. 8-10 show the deformation modes of the cantilever beam which both ANSYS and the developed programs provide the vibration modes corresponding with exact solutions (dash line). For the mode frequencies show that the results obtained from the FEM programs developed in

this work and ANSYS are closer to exact solutions when using more nodes.

Fig. 8 shows that the difference between the computed mode frequency and the exact solution in mode 1 is 18.34% and 18.31% for ANSYS and the developed programs using 561 nodes, respectively. At 2,624 nodes, ANSYS and the developed programs give the results with only 4.51% and 4.50% difference to the exact solution.

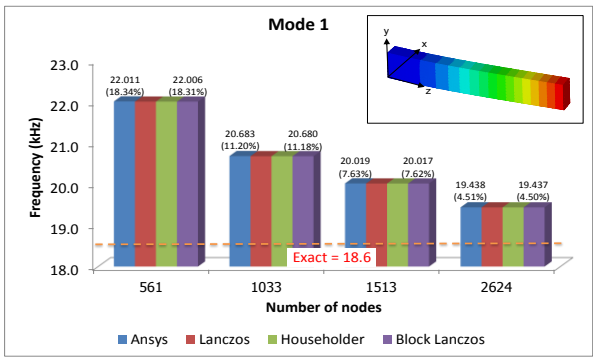

Fig. 8 Comparison of mode frequency values for mode 1 (first bending in x direction)

Figs. 9-10 show that the discrepancy between the computed result and the exact solution in modes 2 and 3 is only ~0.1% and ~1%, respectively, when using 2,624 nodes. At 2,624 nodes, ANSYS and the developed FEM programs give very similar results (less than 0.04% difference).

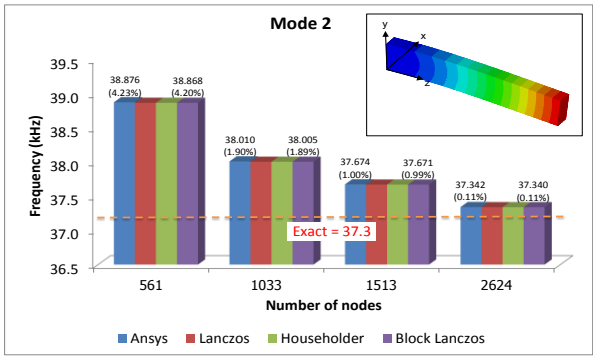

Fig. 9 Comparison of mode frequency values for mode 2 (first bending in y direction)

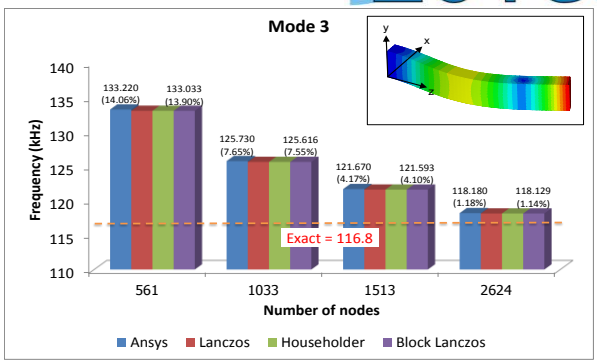

Fig. 10 Comparison of mode frequency values for mode 3 (second bending in x direction)

4.3 Anvil

An anvil is unconstrained as shown in Fig 11 [19]. Material properties of the anvil are Young's Modulus of 2.07×10^{11} N/m^2, Poisson's ratio of 0.3 and density of 7,860 kg/m^3.

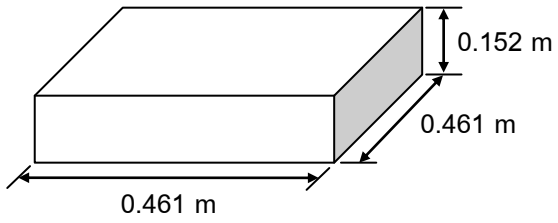

Fig. 11 Geometry of an anvil

Figs. 12-15 show the deformation modes of the anvil which both ANSYS and the developed programs provide the vibration modes corresponding with experimental values (dash line). These figures show that the results obtained from the FEM programs developed in this work and ANSYS are closer to experimental values when node number increases.

Fig. 12 shows that the difference between the computational value and the measured value in mode 1 is 16.72% and 15.94% for ANSYS and the developed programs using 489 nodes, respectively. At 2,461 nodes, ANSYS and the developed programs give the results with only ~5% difference to the measured value.

Figs. 13-15 show that the discrepancy between the computed result and the measured values in modes 2 to 4 is less than 5%, respectively, when using 2,461 nodes. At 2,461 nodes, ANSYS and the developed FEM programs give very similar results (less than 0.5% difference).

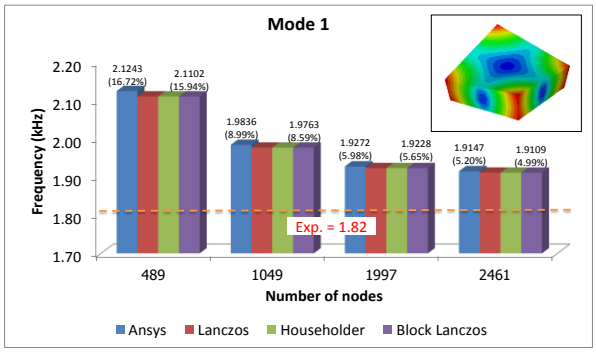

Fig. 12 Comparison of mode frequency values for mode 1 (twist mode)

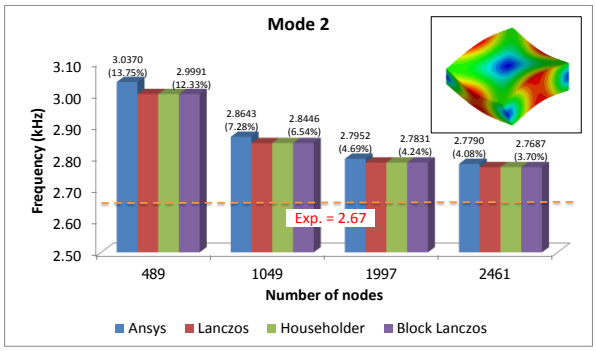

Fig. 13 Comparison of mode frequency values for mode 2 (saddle mode)

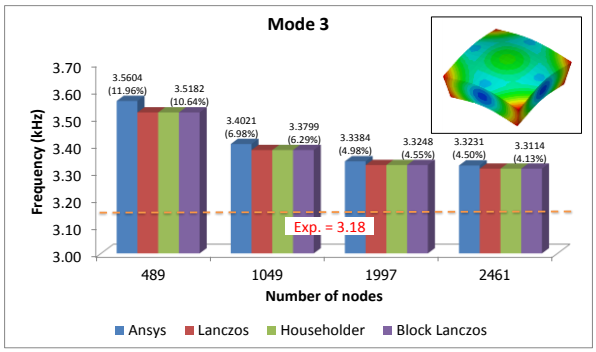

Fig. 14 Comparison of mode frequency values for mode 3 (umbrella mode)

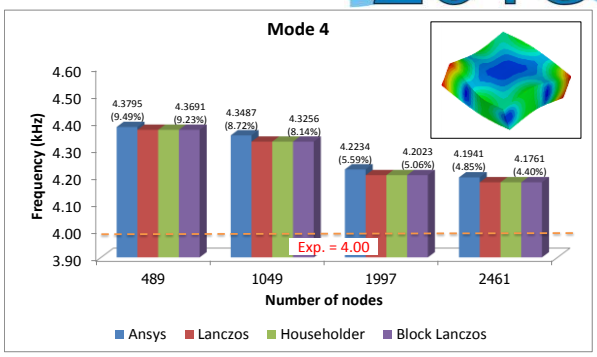

Fig. 15 Comparison of mode frequency values for mode 4 (in-plane shear mode)

5. Conclusions

The development of 3D FEM computer software for free vibration analysis without damping have been demonstrated. The three FEM programs use three tridiagonalization methods, which are Lanczos, Householder and block Lanczos. The computational performances of the FEM computer programs have been evaluated by several problems. The results suggest that the Householder and the block Lanczos methods are more suitable for free vibration analysis than the Lanczos method.

6. Acknowledgements

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