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Stabilizing the Block Lanczos Algorithm for 3D Finite Element Analysis of Free Vibration

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Abstract

The block Lanczos method, an improvement of the classical Lanczos method, has been known as an effective tool to tridiagonalize real symmetric matrices by exploiting memory hierarchies. However, the matrices may lose the orthogonality property after being tridiagonalized, resulting in the presence of rounding errors. Previous research has proposed a numerical scheme to detect and fix the loss of orthogonality during the tridiagonalization process for a symmetric matrix in general eigenvalue problems. In the present study, this detection scheme has been applied to the matrices obtained from 3D finite element derivation for free vibration problem. The results indicate that the numerical solutions obtained from the use of the detection scheme are more accurate than those obtained from the block Lanczos technique alone.

***Keywords**:* Finite element; Eigenvalue problem; Free vibration; Block Lanczos; Orthogonality

1. Introduction

Free vibration analysis is essentially about determining natural frequency of a mechanical system. Design engineers use this calculated result to design structures such that their working frequency interval is sufficiently far from their natural frequency. For 3D complex mechanical structures, free vibration analysis is commonly done by finite element method (FEM).

Free vibration analysis without damping is an eigenvalue problem. Generally, large-scale eigenvalue problems can be solved with the QR/QL algorithm [1], but it requires large CPU time and computer memory. In order to increase computational efficiency, the dense FEM matrix can be stored in the tridiagonal symmetric form before being used for eigenvalue calculation.

One effective method for tridiagonalizing real symmetric matrices is the Lanczos method [2]. The method has been modified to an improved version called the block Lanczos method [3] by exploiting the memory hierarchies. However, matrices may lose orthogonality property after the tridiagonalization process of both the Lanczos and the block Lanczos methods, resulting in the presence of rounding errors. In 2005, Qiao et al.

[3] have proposed a numerical technique to detect the loss of orthogonality and to stabilize the block Lanczos method for general eigenvalue problems. In this paper, this technique has been employed to stabilize the block Lanczos method for free vibration problem. The effectiveness of the stabilizing scheme were also investigated by four numerical problems.

2. The Block Lanczos Method

The FEM equation for free vibration problem can be written as

$$(K-\lambda M)u=0 \qquad (1)$$

where K is a stiffness matrix and M is a mass matrix. Eq. (1) can be transformed to a standard form of the eigenvalue problem as given by:

$$(A-\lambda I)\psi=0 \qquad (2)$$

In Eq. (2), A is a dense matrix with dimension $n\times n$. The idea of the basic block Lanczos algorithm is to change matrix A into a block tridiagonal matrix J containing small square matrices on the diagonal and on the above and the below subdiagonals as shown in Fig. 1.

$$\begin{bmatrix} D_1 & B_1^T & & 0 & 0 \\ B_1 & D_2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & D_{p-1} & B_{p-1}^T \\ 0 & 0 & & B_{p-1} & D_p \end{bmatrix}$$

Fig. 1 Block tridiagonal matrix J

To compute D_i, where $i=1,...,p$, and B_j, where $j=1,...,p-1$, in the block tridiagonal matrix in Fig. 1, the block Lanczos algorithm [3] is implemented as follows.

Set $Q_0=B_0=0$, where Q_0 and B_0 are matrices with dimension $n\times bs$ and $bs\times bs$, respectively. Here, bs is an arbitrary block size such that $n=p\times bs$ and Q_1 is an arbitrary orthonormal matrix. The iteration of block Lanczos for $i=1,...,p-1$, is

$$D_i=Q_i^T A Q_i \qquad (3)$$

$$R_i=AQ_i-Q_iD_i-Q_{i-1}B_{i-1}^T \qquad (4)$$

Decompose R_i by the QR algorithm leads to:

$$Q_{i+1}B_i=R_i \qquad (5)$$

Therefore:

$$D_p=Q_p^T A Q_p \qquad (6)$$

Once matrix J is constructed, it can be transformed into a tridiagonal matrix T as shown in Fig. 2.

$$\begin{bmatrix} \alpha_1 & \beta_1 & & 0 & 0 \\ \beta_1 & \alpha_2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \alpha_{p-1} & \beta_{p-1} \\ 0 & 0 & & \beta_{p-1} & \alpha_p \end{bmatrix}$$

Fig. 2 Tridiagonal matrix T

To compute α_i and β_i in Fig. 2, the tridiagonalization algorithm [3] is applied as follows.

Set $\varpi_0 = \beta_0 = 0$, where ϖ_i and $b = \left[1, 0, \ldots, 0\right]$ are column vectors with dimension $n \times 1$. Then, we compute:

$$\varpi_1 = \frac{b}{\|b\|_2} \tag{7}$$

The iteration loop for $i = 1, ..., p,$ is

$$\alpha_i = \varpi_i^T J \varpi_i \tag{8}$$

$$r_i = J\varpi_i - \alpha_i \varpi_i - \beta_{i-1}\varpi_{i-1} \tag{9}$$

$$\beta_i = \|r_i\|_2 \tag{10}$$

If $\beta_i = 0$, stop the iteration and compute:

$$\varpi_{i+1} = \frac{r_i}{\beta_i} \tag{11}$$

3. The Stabilizing Scheme

The block Lanczos algorithm may lose orthogonality when computing matrix Q_i at each iteration. To detect the loss of orthogonality, the componentwise detection [3] with modification to apply to real symmetric matrices is being considered.

At each iteration, if the rounding error term F_j is added to Eq. (5), the formulation is given by

$$Q_{j+1}B_j + F_j = R_i \tag{12}$$

and

$$R_i = AQ_j - Q_j D_j - Q_{j-1}B_{j-1}^T \tag{13}$$

By moving the term F_j to the right hand side, the transformation is

$$Q_{j+1}B_j = AQ_j - Q_j D_j - Q_{j-1}D_{j-1}^T - F_j \tag{14}$$

and

$$Q_{k+1}B_k = AQ_k - Q_k D_k - Q_{k-1}D_{k-1}^T - F_k \tag{15}$$

Premultiplying the above two equations with Q_k^T and Q_j^T, respectively, and let $W_{k,j} = Q_k^T Q_j$, the equations become

$$W_{k,j+1}B_j = Q_k^T AQ_j - W_{k,j}D_j - W_{k,j-1}D_{j-1}^T - Q_k^T F_j \tag{16}$$

and

$$W_{j,k+1}B_k = Q_j^T AQ_k - W_{j,k}D_k - W_{j,k-1}D_{k-1}^T - Q_j^T F_k \tag{17}$$

Form Eq. (17), the transformation is:

$$Q_j^T AQ_k = W_{j,k+1}B_k + W_{j,k}D_k + W_{j,k-1}D_{k-1}^T + Q_j^T F_k \tag{18}$$

Because the transpose of $Q_j^T AQ_k$ is $Q_k^T AQ_j$ in Eq. (16), and $W_{k,j}^T = W_{j,k}$, substituting $Q_k^T AQ_j$ in Eq. (16) with Eq. (18), we get

$$\begin{aligned} W_{k,j+1}B_j = {} & B_k^T W_{k+1,j} + D_k W_{k,j} + B_{k-1}W_{k-1,j} \\ & - W_{k,j}D_j - W_{k,j}B_{j-1}^T + G_{k,j} \end{aligned} \tag{19}$$

where $W_{j-1,j-1} = W_{j,j} = I$
and $G_{k,j} = F_k^T Q_j - Q_k^T F_j$.

In componentwise detection algorithm, let $w_{x,y}$ be the row *x* and column *y* entry of $W_{k,j+1}$.

If $w_{x,y} > \sqrt{\varepsilon}$ (where ε is an arbitrary round-off error tolerance), the orthogonality is lost. For the term $G_{k,j}$ in Eq. (19), let

$$G_{k,j} = \varepsilon\left(B_k + B_j\right)\left(\Theta_r\right)$$
$$\Theta_r \in N\left(0,0.3\right) \quad (20)$$

which implies that each entry in $B_k + B_j$ is multiplied by a normally-distributed random number with zero mean and 0.3 variance. Also, let $W_{j,j+1}$ be such that

$$W_{j,j+1}B_j = bs \cdot \varepsilon B_1\left(\psi_r\right) \quad (21)$$

where $\psi_r \in N\left(0,0.6\right)$ and *bs* is a block size.

When the absolute value of a component $w_{w,y}$ of $W_{k,j+1}$ exceeds $\sqrt{\varepsilon}$, we find the largest interval $\left[l_k, u_k\right]$ such that for all $k \in \left[l_k, u_k\right]$ and for all $i \in \left[l_k, u_k\right]$, $W_{i,j+1}$ has at least one component larger than a tolerance. Qiao et al. [3] has suggested using the tolerance value of $\varepsilon^{7/8}$.

The loss of orthogonality property in the final tridiagonal iteration stage can be detected using the modified partial orthogonalizaton technique [4]. In this stage, matrix *J* is exploited, and we can obtain the final tridiagonal matrix *T*. The algorithm is similar to the componentwise detection technique explained above where in this case the tolerance is equal to $\varepsilon^{3/4}$.

4. Numerical Evaluations

Four free vibration problems were used to compare the solution accuracy between the basic block Lanczos algorithm and the componentwise block Lanczos algorithm. The problems were also solved by the commercial software MATLAB [5] to obtain referenced solutions. In the result comparison, we assume that simulation results from MATLAB provide accurate solutions.

4.1 Cubic Box

This example is a free vibration analysis of a cubic box [6] which is fixed at one face. The geometry and dimensions of the solid cube are shown in Fig. 3. The cube has Young's modulus (E) of 68.95×10^9 N/m^2, Poisson's ratio (υ) of 0.3, and density (ρ) of 2,560 kg/m^3.

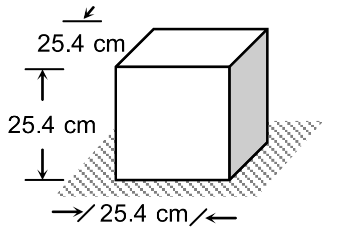

Fig. 3 Geometry of the cubic box

The solutions shown in Fig. 4 were from FEM simulations with 60 nodes and 157 tetrahedral elements. Fig. 4 shows that the results from the basic block Lanczos algorithm have good accuracy for modes 1 to 20, and 135. For modes 21 to 134, the computed natural frequencies from the basic block Lanczos algorithm are not in agreement with those from MATLAB. On the other hand, the orthogonalized block Lanczos method provides accurate solutions for all modes.

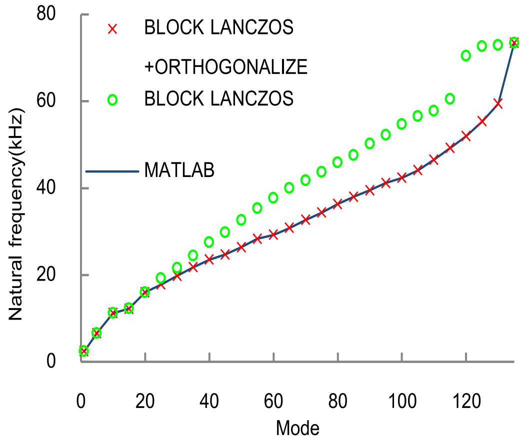

Fig. 4 Comparison of the solutions for the cubic box problem

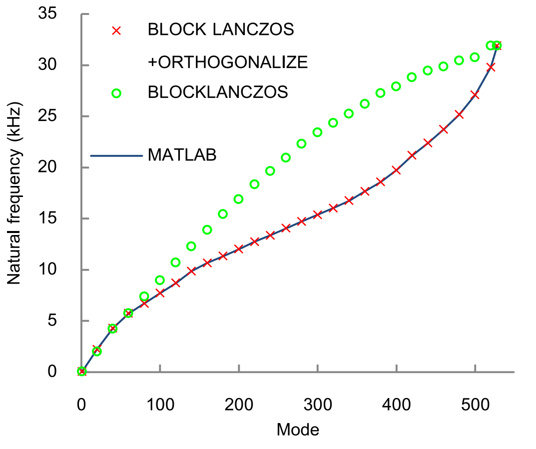

Fig. 6 Comparison of the solutions for the cantilever beam problem

4.2 Cantilever Beam

The cantilever beam [6] in Fig. 5 is fixed at one end. It has Young's modulus (E) of 2.068×10^{11} N/m^2, Poisson's ratio (υ) of 0.3, and density (ρ) of 8,058 kg/m^3.

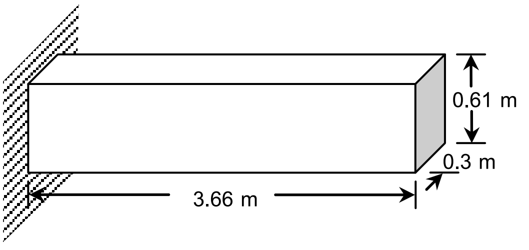

Fig. 5 Geometry of the cantilever beam

Natural frequency for each mode has been computed by FEM with 188 nodes and 520 tetrahedral elements. Fig. 6 shows that the basic block Lanczos algorithm provides accurate solutions only for modes 1 to 14 and 528. In modes 15 to 527, the solutions from the basic block Lanczos algorithm are not in agreement with those from MATLAB. On the other hand , the orthogonalized block Lanczos method provides accurate solutions for all modes.

4.3 Anvil

The free-constrained anvil [7] in Fig. 7 has a dimension of 0.461 m $\times$ 0.461 m $\times$ 0.152 m. It has Young's modulus (E) of 2.07×10^{11} N/m^2, Poisson's ratio (υ) of 0.3, and density (ρ) of 7,869 kg/m^3.

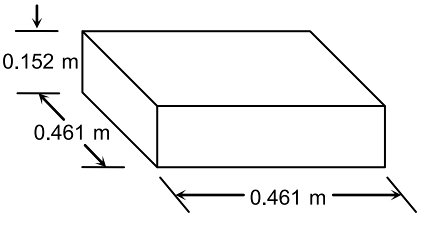

Fig. 7 Geometry of the anvil

FEM simulations were performed using 187 nodes and 561 tetrahedral elements. Fig. 8 shows that the basic block Lanczos algorithm provides accurate solutions in modes 1 to 3 and 100. For the other modes, the solutions from the basic block Lanczos algorithm are not in agreement with those from MATLAB. On the other hand, the orthogonalized block Lanczos method provides accurate solutions for all modes.

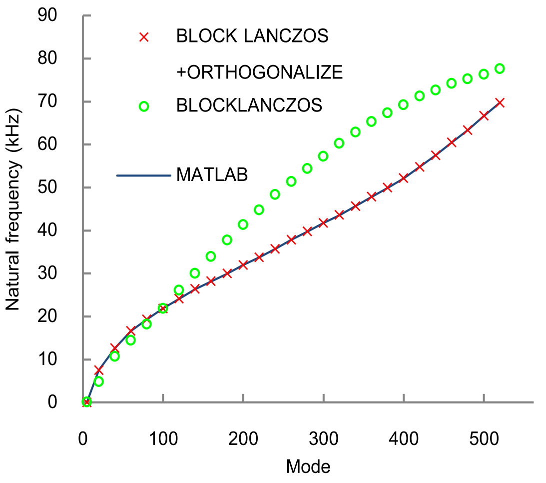

Fig. 8 Comparison of the solutions for the anvil problem

4.4 Turbine Blade

The turbine blade [6] with complex geometry in Fig. 9 is fixed at one end. It has Young's modulus (E) of 2.07×10^{11} N/m^2, Poisson's ratio (υ) of 0.3, and density (ρ) of 7,860 kg/m^3.

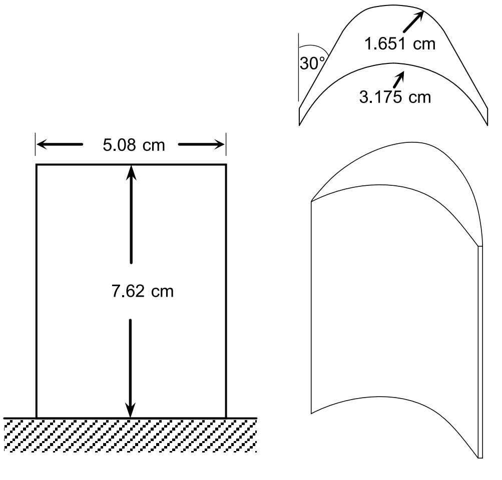

Fig. 9 Geometry of the turbine blade

FEM simulations were performed using 240 nodes and 682 tetrahedral elements. Fig. 10 shows that the basic block Lanczos algorithm provides solutions that are not in agreement with those from MATLAB for all modes. On the other hand, the orthogonalized block Lanczos method and MATLAB provides similar solutions for all modes.

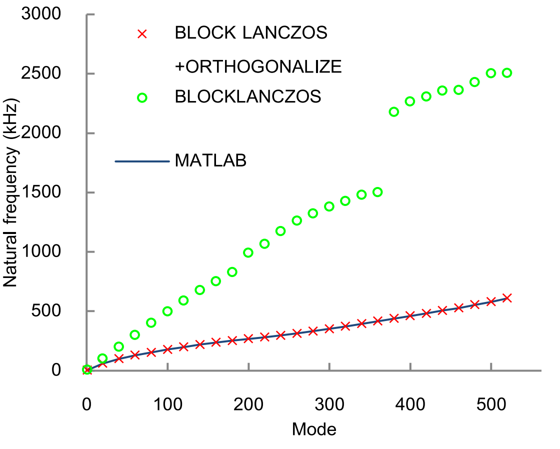

Fig. 10 Comparison of the solutions for the turbine blade problem

6. Conclusions

The application of the detection scheme for detecting and fixing the loss of matrix orthogonality in 3D FEM free vibration problem has been studied. The results indicate that the detection scheme enhances solution accuracy obtained from the FEM analysis using the block Lanczos tridiagonalization method. This is attributed to the preservation of matrix orthogonality which is important to avoid rounding errors.

7. Acknowledgements

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